THE TERNARY DESCRIPTION LANGUAGE AS A FORMALISM FOR THE PARAMETRIC GENERAL SYSTEMS THEORY : PART III

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This part is a continuation of the first and the second parts of my article that were published in the *International Journal of General Systems*, vol. 28 (4-5), pp.351-366; vol. 31 (2), pp.131-151. In the Part III, we deal with the construction of the axiomatic system of the Ternary Description Language (TDL). Axioms and Rules of inference are formulated. On the base of these Axioms and Rules some theorems of TDL are proved. Several system-theoretical laws, which concern the values of systems parameters, are proved as theorems of TDL. Thus the deductive construction of General Systems Theory is made.

Keywords: Syntactical priority; Synonymy; Axioms; Rules of substitution; Rules of replacement; Theorems of the TDL; System–Theoretical Laws.

1. THE BRIEF EXPOSITION OF THE PARTS I AND II

This section serves as an introduction to the Part III. It is a brief outline of what I have argued previously. The short account of the essentials of logical formalism called *Ternary Description Language* (TDL) that intends to be a language of the Parametric General Systems Theory (PGST) follows below.

The TDL is based on three categories: Thing (= object = entity), Property and Relation. The essential feature of our conceptual framework is the contextual character of distinction between the categories. It means that a thing in one context can be a property or relation in another context. Things, properties and relations can be *definite*, *indefinite* and *arbitrary*. We denote the definite object by the symbol t, an indefinite object by the symbol a, an arbitrary object by the symbol A. Formulae

I. t, a, A

are *elementary well-formed formulae* (WFF) of our formalism – TDL. The other types of WFF are formed in the following manner:

II. $(A)A$

– Arbitrary *thing has* an arbitrary *property*. Here we can substitute A for any WFF, and the result of such a substitution is considered WFF, too, e.g., $(A)a$, $(t)a$, $(a)t$, $((A)\,a)\,a$ are WFF. The same is valid for the formulae given below.

III. $A(A)$

– Arbitrary *thing has* an arbitrary *relation*. $a(A)$, $a(t)$, $t(a)$, $a(a(A))$ are special cases of that type of WFF.

IV. $(A^*)A$

– This type of WFF differs from (II) in the direction of the predicate relation. The formula means that an arbitrary *property belongs to* an arbitrary *thing*, e.g.: $(a^*)A$, $(a^*)t$, $(a^*)a$, $(a^*)(a)t$.

V. $A(^*A)$

– An arbitrary *relation realizes on* an arbitrary *thing*, e.g.: $A(^*a)$, $t(^*a)$, $a(^*a)$, $t(a)(^*a)$.

The formulae of the types (II)-(III) may be called *direct* ones, and the formulae of the types (IV)-(V) – *inverse* formulae.

VI. $[A]$

– A formula of this type means what may be called the *conceptual closure* of the formula A. If A expresses a proposition, then $[A]$ denotes the *concept corresponding to* that *proposition*. The conceptual closure of $(A)A$ gives us the formula $[(A)A]$ that is interpreted as "an arbitrary thing possessing an arbitrary property." Similarly $[(A^*)A]$ denotes "an arbitrary property inherent in the arbitrary thing", etc.

The formulae of the type (II)-(V) are *open*, while the formulae with square outermost brackets are *closed*.

VII. $\{A\}$

– Curly brackets have an ancillary character. They are used in the case when the inclusion of one formula into another as a subformula leads to ambiguity.

VIII. A, A

– This type of WFF is a simple list of WFF. We shall call the formulae of such a type *free lists*, because they do not suppose any relation between their components. Note that the order of formulae in a list is ignored. The combinations of symbols t, a and a, t are regarded as one and the same combination.

Nevertheless, the order of symbols is very essential in the other types of WFF. We have seen it in the examples of direct and inverse formulae. The concrete meanings of A and a objects (but not t object) depend on their environment in a formula.

Let us explain this dependence. We distinguish the first – initial, and the second – final parts in every two-member formula listed above. In direct formulae initial parts are included in parentheses. They denote things. In inverse formulae the initial parts are placed outside of parentheses. They denote properties and relations. An indefinite thing a, when it is placed on the initial part of an open formula, has an unlimited range of indefiniteness. However, when a appears in the final part of a formula, it has a restricted indefiniteness. In $\{(a)a\}$ the second a is an indefinite object, but is also a property of the first a.

In the case of complicated formulae, which consist of non-elementary subformulae, we can also (recursively) define the initial part, i.e. the *beginning* of the formula. The indefiniteness placed on the beginning of an open formula will be called *initial*, while the indefiniteness restricted by the sentence context – *contextual*. In the closed formulae the indefiniteness can be contextual even on the initial place, e.g. $[(a)t]$.

If there are two or more occurrences of the symbols a or A in the same formula, this does not mean that they necessarily denote the same object. On the other hand, different subformulae can denote one and the same thing (just like in natural languages).

In cases when it is known that various occurrences of the same or different subformulae denote *the same object*, this fact should be expressed with the aid of additional symbols in front of these subformulae. There is no need to include these symbols in the list of WFF's types since formulae with identifying symbols can be formally defined through WFF listed above, as shown in our previous paper. We have constructed our formal definition of identity based on a well known principle that was formulated by Aristotle and is usually called Leibniz's principle: "What is said about one thing should be said about the other".

It is important to take the direction of the identification into account because without distinguishing directions of identification we could not distinguish the operations of synthesis and analysis.

We use the small Latin letter j (italic jay) to denote an object with which the identification is being carried out: jA. Jay bold-faced letter in front of the formula denotes an object being identified: $\mathrm{j}A$. The assertion of the identity of any object to any object will have the form: $\mathrm{j}A\, jA$. In particular: $\{\ \mathrm{j}a\ ja\ \}$, $\{\ \mathrm{j}A\ ja\ \}$.

Formulae with jay operators are analogous to the identity relation that is represented by the "=" symbol, e.g. in algebra: $(a+b)^2 = a^2+2ab+b^2$. But in algebra we can observe other type of identifications, which are related to separate terms in the formula. E.g., both occurrences of a in $a^2+2ab+b^2$ denote identical numbers. Algebra does not require a special symbol for that kind of identity because of the assumption that it is expressed by the identity of the forms of symbols. But in our case the same symbol a (or A) can denote different objects in different occurrences. Therefore if those objects are factually identical, it is necessary to use special marks for the corresponding occurrences of terms.

In this paper we shall restrict ourselves to the case of the *undirected* identity of terms, denoted by the Greek letter ι (iota) in front of formulae representing identified objects. It can be shown that iota operators can be formally defined through jay operators. Examples of iota operators usage: $(\iota A)\iota A$, $(\iota A^*)\iota A$. Not just one, but many different identifications can occur in the same formula. In this case, several types of iota operators are used. In order to obtain the necessary variety of these operators the letter iota can have various subscripts, be doubled, tripled, etc. E.g.: $([(\iota A)\iota\iota A])[(\iota A)\iota\iota A]$, $([(\iota_8 A)\iota_{13}A])[(\iota_8 A)\iota_{13}A]$.

Let us now turn to problems of GST. Various authors give different definitions of the system's concept. Some are too "broad", e.g. "Systems are sets of objects with some relations between them". We can express such a definition with the aid of the following TDL formula:

$$(\ \iota A\ )\mathrm{S}\ =_{def}\ \ a(\iota A) \qquad (1)$$

Another kind of definition is connected to the specification of the "system-making" relation. Systems are defined as "Complexes of interacting elements", "Complexes of interconnected elements", or "Ordered sets of elements", etc. Such definitions can be expressed with the help of the TDL formula:

$$(\ \iota A\ )\mathrm{S}\ =_{def}\ [\ (a)\ t\ ]\ (\iota A) \qquad (2)$$

All definitions of the form (2), where t is a concrete property, are too "narrow", because attempts to find t that is appropriate for *any* system research fail. Every concrete t has its defects. From our point of view the solution of the problem is in the permission to *change the* *interpretation of* t. Instead of concrete t we will refer to t in general. In this case we can take the above formula not as a scheme of definitions, but as a pure definition: "*A system is an arbitrary thing in which a relation having a definite property is realized*".

We can rewrite formula (2) in equivalent but more compact form:

$$(\iota A)\mathrm{S} =_{def} ([a(*\iota A)])\, t \qquad (3)$$

The majority of definitions given in the literature can be considered particular cases of our definition (3). Nevertheless there are some exceptions. At times the role of definite t in the system definition moves to a relation (see Rapoport, 1966). In such cases we should have:

$$(\iota A)\mathrm{S} =_{def} (\iota A)[t(a)] \qquad (4)$$

$$(\iota A)\mathrm{S} =_{def} t([(\iota A*)\, a]) \qquad (5)$$

In words: "*A system is an arbitrary thing in which properties having a definite relation between them are realized*". This definition is dual to the previous one in respect to transformation "property – relation."

The next problem (to which the previous part II of my article was mainly dedicated) is the definition of some subsidiary concepts of the TDL.

First, we implement several types of implications. The first, called *attributive*, is defined with the aid of the following formula:

$$\{\iota A \Rightarrow \iota\iota A\} =_{def} \mathrm{j}\iota A\, j\, [(a)\iota\iota A] \qquad (6)$$

Here, the definiens expresses identification of the object denoted by ιA with some object possessing properties expressed by $\iota\iota A$. This holds in categorical sentences expressed with the aid of the copulative verb "is". When we say that a square is a rectangle, we mean that a square is identical to some object endowed with the properties of a rectangle.

If objects of the list are considered separately, not relating to each other, then, even without denying the presence of a relation between them, we are dealing with an ordinary, simple *free list*.

If the list somehow relates objects to each other, then, in this case we shall call it *related* and denote it by $\{\iota A \bullet \iota\iota A\}$. We can give the following definition of such a list:

$$\{\iota A \bullet \iota\iota A\} =_{def} [(\iota\iota\iota A)\{[A(*\iota A, \iota\iota A)] \Rightarrow [a(*\iota\iota\iota A)]\}] \qquad (7)$$

The meaning of this definition resides in the fact that for a related list $\iota A \bullet \iota\iota A$ there is such an object $\iota\iota\iota A$ that any relation to an object ιA or $\iota\iota A$ will simultaneously be a relation to the object $\iota\iota\iota A$. Consequently, through a certain intermediate object – $\iota\iota\iota A$, the objects ιA and $\iota\iota A$ turn out to be related to each other.

Using the concept of a related list, let us define the second type of implication, which we shall denote by the symbol $\supset$:

$$\{\iota A \supset \iota\iota A\} =_{def} \mathrm{j}\iota A\, j\{\iota\iota A \bullet a\} \qquad (8)$$

The implication defined by this formula is called *mereological*. E.g.: Ukraine mereologically implies Odessa.

The third type of implication is *relational*. It is analogous to the attributive implication, but the role of attribute in it is played by a relation. Let us denote the relational type of implication by the symbol $\succ$. We shall define it with the help of the next formula:

$$\{ \iota A \succ \iota\iota A \} =_{def} \ \mathrm{j}\iota A\, j[\iota\iota A(a)] \tag{9}$$

E.g. a map of a city (ιA) implies relations between streets of that city ($\iota\iota A$). An object ιA is identical to some object with relations $\iota\iota A$.

We can generalize the three types of implications defined above in the concept of *neutral* implication. We use a simple arrow to denote it:

$$\{ \iota A \to \iota\iota A \} =_{def} ([A(*\iota A \cdot \iota\iota A)])\{[(\iota A \Rightarrow \iota\iota A^*)\iota\iota\iota A], [(\iota A \supset \iota\iota A^*)\iota\iota\iota A], [(\iota A \succ \iota\iota A^*)\iota\iota\iota A]\} \tag{10}$$

According to this definition, neutral implication is a relation between a list of related objects, which possesses an arbitrary property that can be ascribed to the attributive, mereological, and relational implications.

Note that in defining implications we did not make any use of valent values – Truth and Falsity. This grants us the possibility for inverting the problem, i.e. for defining valent values through an implication introduced independently of them.

Truth and Falsity are regarded as some properties of TDL formulae, not only for open formulae but closed as well. In the previous part (Part II) of this article the "true" formula, signed as (A)T, and the "false" formula (A)F, were defined.

Based on the concepts of T, F and neutral implication $\to$, there appears the possibility to define the operation of disjunction:

$$\{ \iota A \vee \iota\iota A \} =_{def} \{ \{(\iota A)\mathrm{F} \to (\iota\iota A)\mathrm{T} \} \cdot \{(\iota\iota A)\mathrm{F} \to (\iota A)\mathrm{T}\} \} \tag{11}$$

Note, that the above property F was defined to denote so-called *contrary* falsity. This kind of falsity can be applied directly to any formula. A derivative kind is *contradictory* falsity (contradictory negation), which is denoted by the symbol n and can be applied only to the formulae which have already been valent. It was defined in part II with the help of formal definitions like the following:

$$((A)\mathrm{T})n =_{def} (a)\mathrm{F} \tag{12}$$

$$((A)\mathrm{F})n =_{def} (a)\mathrm{T} \tag{13}$$

Taking into account the contradictory falsity, we obtain three types of valent endings of the TDL formulae: T, F and n. In addition, it is also possible to introduce the fourth valency, when a formula has the valencies T and F simultaneously. For example, it is true that (a)T, and also true that (a)F. In general form it can be expressed as: $(a)\{\mathrm{T,F}\}$. Formulae that have the values T and F simultaneously may be called *ambivalent* ones. It is possible to combine the ambivalence and contradictory negation. For example, if there is $((A)\mathrm{T})n$ and $((A)\mathrm{F})n$ simultaneously, we can express it in the united formula: $((A)\{\mathrm{T,F}\})n$.

Valencies, which were considered above, may be called *definite* ones. *Indefinite valency* may be ascribed to non-valent formulae, if they can have valency in principle. In addition to this *quasidefinite* valency can be defined, when one valency, T or F is known and other is possible. A quasitrue formula is a true or ambivalent one. Let us denote such a valency by $\{\mathrm{T},\ \}$. Correspondingly quasifalsity is denoted as $\{\mathrm{F},\ \}$.

It is supposed that the definiens and definiendum are true simultaneously. Therefore if the definiendum is true, the definiens is true also. In this case the valent sign T may be omitted both after the definiendum and definiens.

Because in this paper implications of any kind will be seldom used as not-valent formulae, while true implications will be used very often, it is convenient to agree here that the absence of the valency sign after an implication denotes the mark T. If an implication lacks the valency T, we will always follow it with one of these symbols: F, T)n, F)n , {T,F} , {T, } , {F, }.

To formalize the system parameters values we need to define some specific “objects” by application of previously defined operations to the basic objects a, A, t of the TDL.

Let $\iota A'$ be an indefinite object which is *different* from ιA. Formally:

$$\iota A' =_{def} [(\iota\iota a)\{ (\{ \iota\iota a \supset \iota A \} \bullet \{ \iota A \supset \iota\iota a \})\mathrm{F} \}] \tag{14}$$

In accordance with the accepted notations, the definiens of (14) means that it is impossible to have both implications simultaneously. And this impossibility is the property of $\iota\iota a$.

An indefinite *subobject* – $\overset{\cup}{\iota A}$ – has a definition:

$$\overset{\cup}{\iota A} =_{def} [(\iota A')\{ \iota A \supset \iota A'\}] \tag{15}$$

An indefinite *superobject* – $\overset{\Delta}{\iota A}$ – is defined as:

$$\overset{\Delta}{\iota A} =_{def} [(\iota A')\{ \iota A' \supset \iota A \}] \tag{16}$$

An indefinite disparate – $\overset{\circ}{\iota A}$ – is defined as an object, different from ιA, which possesses two properties simultaneously: it is neither a subobject nor a superobject. Formally:

$$\overset{\circ}{\iota A} =_{def} [(\iota A')\{ \{((\iota A'^{*})\overset{\cup}{\iota A})\mathrm{F}\} \bullet \{((\iota A'^{*})\overset{\Delta}{\iota A})\mathrm{F}\} \}] \tag{17}$$

Finally, we will give the definition of the *limited* object – $L\iota A$. The idea of such an object is that the addition of something to an object ιA does not usually mean that the entity of ιA will become different. For example, if we put gloves on the cat in boots it will still remain the cat in boots, in spite of the fact that gloves are not a part of boots. The situation is different if we have a cat *in boots only*. In order to emphasize this point, we should say that every addition that preserves an entity of “the only ιA” (signed by $L\iota A$) must occur in $L\iota A$:

$$L\iota A =_{def} [(\iota A)\{ \{ \{ \iota A \bullet \iota\iota A \} \Rightarrow \iota A \} \supset \{ \iota A \supset \iota\iota A \} \}] \tag{18}$$

We can also obtain objects $\iota a'$, $\overset{\cup}{\iota a}$, $\overset{\Delta}{\iota a}$, $\overset{\circ}{\iota a}$, $L\iota a$, t', $\overset{\cup}{t}$, $\overset{\Delta}{t}$, $\overset{\circ}{t}$, Lt, by substitution of the corresponding symbols instead of A into the definitions given above.

Using the notions mentioned above, the formal definitions for the values of binary systems parameters were proposed in the part II of this article. Some of these formalizations will be exemplified below.

(ιA)Structurally open system $=_{def}$

$$(\iota A)\{\ \{(\ [\iota\iota\, a\ (*\iota A)]\ )t\} \bullet \{(\ [\iota\iota\,\overset{\Delta}{a}(*\iota A)]\ )t\}\} \tag{19}$$

(ιA)Structural-non-point system $=_{def}$

$$(\iota A)\{\ \{(\ [\iota\iota\, a\ (*\iota A)]\ )t\} \bullet \{(\ [\iota\iota a'(*\iota A)]\ )t\}\ \} \tag{20}$$

Note that the above parameter had more complicated (but, as it can be shown, equivalent) definition in the part II of this article.

(ιA)Homeomery system $=_{def}$

$$(\iota A)\{\ \{(\ [\iota\iota\, a(*\iota A)]\ )t\} \bullet \{(\ [\iota\iota\, a(*[(A)\iota\overset{\cup}{A}\ ]\ )])t\}\ \} \tag{21}$$

(ιA)Non-minimal system $=_{def}$

$$(\iota A)\{\{(\ [\iota\iota\, a(*\iota A)]\ )t\} \bullet\{(\ [\iota\iota\, a(*\iota\overset{\cup}{A})]\ )t\}\} \tag{22}$$

(ιA)Internal-centric system $=_{def}$

$$(\iota A)\{\{([a(*\iota A)])t\} \bullet\{\iota\iota\ \{\iota\overset{\cup}{A}\ \} \bullet\{[A(*\iota A)] \Rightarrow [a(*\iota\iota\ \{\iota\overset{\cup}{A}\ \})]\}\}\} \tag{23}$$

(ιA)External-centric system $=_{def}$

$$(\iota A)\{\{([a(*\iota A\bullet\iota\mathring{A})])t\} \bullet\{\iota\iota\{\iota\mathring{A}\} \bullet\{[A(*\iota A)] \Rightarrow [a(*\iota\iota\{\iota\mathring{A}\})]\}\}\} \tag{24}$$

(ιA)Centric system $=_{def}$

$$(\iota A)\{\{([a(*\iota A)])t\} \bullet \{\iota\iota\, a \bullet\{[A(*\iota A)] \Rightarrow [a(*\iota\iota\, a)]\}\}\} \tag{25}$$

(ιA)Non-immanent system $=_{def}$ $(\iota A)\{(\ [a(*\iota A\ \bullet\iota\mathring{A}\ )]\ )t\}$ (26)

(ιA)Non-elementary system $=_{def}$

$$(\iota A)\{\{(\ [\ a(*\iota A)]\ )t\}\bullet\{(\ [\ a(*\iota\overset{\cup}{A}\ )]\ )t\ \}\} \tag{27}$$

(ιA)Non-unique system $=_{def}$

$$(\iota A)\{\{(\ [\iota\iota a(*\iota A)]\ )t\} \bullet \{(\ [\iota\iota a(*\iota A')]\ )t\ \}\} \tag{28}$$

(ιA) Substratum-open system $=_{def}$

$$(\iota A)\{\{(\ [\iota\iota a(*\iota A)]\ )t\} \bullet \{(\ [\iota\iota a(*\iota\overset{\Delta}{A})]\ )t\ \}\} \tag{29}$$

(ιA)Totalitarian system $=_{def}$

$$(\iota A)\{\{(\ [\ a(*\iota A)]\ )t\} \bullet\{\ t \rightarrow [A(*\iota A)]\}\} \tag{30}$$

(ιA)Structurally non-variable system $=_{def}$

$$(\iota A)\{\{([\iota\iota a\,(*\iota A)])t\}\bullet\{[A(*\iota A)] \Rightarrow \iota\iota a\}\} \tag{31}$$

(ιA)Rigid system $=_{def}$ $(\iota A)\{\{([\iota\iota a\,(*\iota A)])t\}\bullet\{t \to \iota\iota a\}\}$ (32)

(ιA) Substratum heterogeneous system $=_{def}$

$$(\iota A)\{\{([(*\iota A)])t\}\bullet\{[(\iota \breve{A})\iota\iota A]\bullet[(\iota \breve{A})\iota\iota \mathring{A}]\}\} \tag{33}$$

Note that the definition of substratum heterogeneous system in the part II is inexact because there occurs $\iota\iota A'$ instead of $\iota\iota \mathring{A}$ in the end of the formula.

2. AXIOMS AND RULES OF THE TERNARY DESCRIPTION LANGUAGE

It is necessary to have some axioms and rules of inference for establishing relations between the values of various systems parameters. These relations will be the laws of the Parametric GST.

2.1. AXIOMS

2.1.1. *The Principle of Syntactical Priority.*

Every axiom is a relation between some elementary formulae. How can we believe in the truth of axioms? There are two ways. First, the truth of an axiom is a consequence of the meanings of the elementary formulae that occur in it. Here the *priority of semantics* has place. This priority is natural if we assume that meanings of the elementary formulae are strictly determined irrespective of the relations in which these formulae are occasionally entered. For example, natural numbers have single meanings and the relations between numbers depend on their meanings not the other way. Within the Language of Ternary Description the objects of this kind are t and Lt (and some other, which can be derived from these two). But the meanings of the other objects depend on the context, i.e. on the relations in which these objects are entered being sub-formulae of the widest formulae. Thus, as explained in the part II, the sphere of the uncertainty of the object a depends on the initial or not initial place of this occurrence of a in the formula. The meaning of A in the formula $A \Rightarrow a$ is other than in the formula $[(A)t]$. In the second formula the sphere of the arbitrariness of A is narrowed due to the demand of having the property t. Therefore, in general for the determination of the meaning of elementary formulae in TDL it is necessary to know the current relations that bind them.

So we arrive at the idea of *syntactical priority* in the determination of the meanings of axioms. Syntactical relations are postulated *before* the determination of the meanings of elementary formulae. These meanings are the functions of the postulated syntactical relations fixed in axioms. In other words, to interpret the meaning of a concrete occurrence of an elementary formula in any chosen axiom one must start from an axiom as a whole and compare its interpretation (obtained by substitution of a for "*some*"; t for "*fixed*"; $[(t^*)A]$ for "*an arbitrary property of the fixed thing*"; etc.) with one's intuition of the assertion expressed by the axiom. The meaning of the occurrence mentioned above is the result of the interpretation of a whole axiom.

If the syntactical relation $A \supset a$ will be postulated as axiom, then the interpretation of A would be extracted from the meaning of a whole phrase: "*Any object includes something in it*". If, in turn, the relation $[(A)t] \rightarrow t$ will be an axiom, the meaning of A would be derived from the phrase "*Any object possessing the fixed object as a property somehow implies this fixed object*", and this meaning will be obviously different from the previously stated meaning.

Thus, the second way to believe in axioms supposes that the reader adapt the interpretations of the elementary formulae that occur in an axiom to the truth of this axiom as a whole. This adaptation is partially limited by the initial interpretations assigned to the elementary formulae (e.g., we can't interpret an occurrence of A as "fixed object", but ought to use "any object"; nevertheless the context of the given occurrence applies certain limitations to this "any object").

Before starting the formulations of axioms, let us adduce one abbreviation for formulae representation. Consider well formed formulae schemes II – V. Let us use the metalanguage symbol K to denote an arbitrary formulae scheme of these four. In other words, if you will see below an expression like $\{\iota A \text{ K } \iota\iota A\}$, then all that is said there about this expression is true for every of the following WFF's: $\{(\iota A)\iota\iota A\}$, $\{(\iota\iota A^*)\iota A\}$, $\{\iota\iota A(\iota A)\}$, $\{\iota A(^*\iota\iota A)\}$. (Recall that in part II of this series two analogous to K, symbols R and Q, were introduced, where R abbreviates 8 types of WFF, and Q – 16 types of WFF.)

Note that the object ιA, standing before K in $\{\iota A \text{ K } \iota\iota A\}$, occupies the *initial* position in each of the four considered WFF. Note also that $\{\iota A \text{ K } \iota\iota A\}$ denotes *open* formulae. The structure of corresponding closed formulae can be expressed by $[\iota A \text{ K } \iota\iota A]$. Let us stipulate also that if the symbol K is repeated in various parts of the same formula (or rule, or definition), then K denotes the same WFF-scheme in these repeated occurrences (this feature is the same that characterized R and Q abbreviations).

Taking into account these features of the symbol K we will use K in the axioms of the TDL given below.

2.1.2. *Axioms of synonymy*.

Let us start with axioms of auxiliary character, the implementations of which will help us to simplify other axioms. These are the *axioms of synonymy*.

The synonyms are expressions with the same meaning. We shall denote the relation of synonymy by a metalanguage sign $=_{syn}$. This sign is analogous to the sign of definition $=_{def}$. But while the definition constitutes the meaning of its left part (definiendum) as identical to a meaning of its right part (definiens), synonymy identifies meanings that can be constituted independently one from another.

It is possible to give a formal definition of the synonymy relation:

$$\{\iota A =_{syn} \iota\iota A\} =_{def} \{\iota\iota\iota\{\iota A\}, \iota\iota\iota\{\iota\iota A\}\} \quad (34)$$

So the synonymy relation $=_{syn}$ can be regarded as the WFF.

The first group of axioms considers the *synonymy of free lists*:

$$\{\{\iota A, \iota\iota A\}\ \mathrm{K}\ \iota\iota\iota A\ \} =_{syn} \{\{\iota A\ \mathrm{K}\ \iota\iota\iota A\}, \{\iota\iota A\ \mathrm{K}\ \iota\iota\iota A\}\ \} \tag{35}$$

According to what was said earlier about the symbol K, there are factually four axioms included in (35), one of them is:

$$\{(\iota A, \iota\iota A)\iota\iota\iota A\ \} =_{syn} \{\{(\iota A)\iota\iota\iota A\}, \{(\iota\iota A)\iota\iota\iota A\}\ \} \tag{36}$$

The next axioms are analogous to those of (35), but consider the *closed* formulae:

$$[\{\iota A, \iota\iota A\}\ \mathrm{K}\ \iota\iota\iota A\ ] =_{syn} \{\ [\iota A\ \mathrm{K}\ \iota\iota\iota A]\ , [\iota\iota A\ \mathrm{K}\ \iota\iota\iota A]\ \} \tag{37}$$

The example of a concrete axiom here may be:

$$[(\iota A, \iota\iota A^*)\iota\iota\iota A\ ] =_{syn} \{\ [(\iota A^*)\iota\iota\iota A]\ , [(\iota\iota A^*)\iota\iota\iota A]\ \} \tag{38}$$

$$\{\iota\iota\iota A\ \mathrm{K}\ \{\iota A, \iota\iota A\}\ \} =_{syn} \{\{\iota\iota\iota A\ \mathrm{K}\ \iota A\}, \{\iota\iota\iota A\ \mathrm{K}\ \iota\iota A\}\ \} \tag{39}$$

For example:

$$\{\iota\iota\iota A(\iota A, \iota\iota A)\ \} =_{syn} \{\{\iota\iota\iota A(\iota A)\}, \{\iota\iota\iota A(\iota\iota A)\}\ \} \tag{40}$$

$$[\iota\iota\iota A\ \mathrm{K}\ \{\iota A, \iota\iota A\}\ ] =_{syn} \{\ [\iota\iota\iota A\ \mathrm{K}\ \iota A]\ , [\iota\iota\iota A\ \mathrm{K}\ \iota\iota A]\ \} \tag{41}$$

For example:

$$[\iota\iota\iota A(*\iota A, \iota\iota A)] =_{syn} \{\ [\iota\iota\iota A(*\iota A)]\ , [\iota\iota\iota A(*\iota\iota A)]\ \} \tag{42}$$

$$\{\{\iota A, \iota\iota A\}\ \mathrm{K}\ \{\iota\iota\iota A, \iota\iota\iota\iota A\}\ \} =_{syn}$$
$$\{\{\iota A\ \mathrm{K}\ \iota\iota\iota A\}, \{\iota\iota A\ \mathrm{K}\ \iota\iota\iota A\}, \{\iota A\ \mathrm{K}\ \iota\iota\iota\iota A\}, \{\iota\iota A\ \mathrm{K}\ \iota\iota\iota\iota A\}\} \tag{43}$$

The synonymy relation may appear as a subformula included in another formula, e.g. in implication:

$$(\iota A)\iota\iota A \rightarrow \{\ \iota A =_{syn} [(\iota A)\iota\iota A]\ \ \} \tag{44}$$

$$(\iota\iota A^*)\iota A \rightarrow \{\ \iota A =_{syn} [(\iota\iota A^*)\iota A]\ \ \} \tag{45}$$

$$\iota\iota A(\iota A) \rightarrow \{\ \iota A =_{syn} [\iota\iota A(\iota A)]\ \ \} \tag{46}$$

$$\iota A(*\iota\iota A) \rightarrow \{\ \iota A =_{syn} [\iota A(*\iota\iota A)]\ \ \} \tag{47}$$

The last group of axioms in this section postulates the synonymy relations between the neutral and the valent variants of certain TDL formulae:

$$t =_{syn} [(\ t\ )\mathrm{T}] \tag{48}$$

$$Lt =_{syn} [(Lt\ )\mathrm{T}] \tag{49}$$

$$\overset{\cup}{t} =_{syn} [(\overset{\cup}{t})\mathrm{T}] \tag{50}$$

$$a =_{syn} [(a)\{\mathrm{T, F}\}] \tag{51}$$

$$t' =_{syn} [(t')\{\mathrm{T, F}\}] \tag{52}$$

$$\overset{\Delta}{t} =_{syn} [(\overset{\Delta}{t})\{\mathrm{T, F}\}] \tag{53}$$

$$\mathring{t} =_{syn} [(\mathring{t})\{\mathrm{T, F}\}] \tag{54}$$

Let us arrange the other axioms into groups according to the presence of certain structural elements in them.

2.1.3. *Axioms that are valent formulae and that contain neither implications nor iota operators.*

The first axiom in this group is formulated based on the interpretation of the "definite object" *t*. The fixing of *t* must precede to any application of TDL. Within the Parametrical General Systems Theory, *t*, usually plays the role of the system's concept. It is advisable (for avoiding paradoxes) to have identify the concept in the variety of real objects. So the first is the following:

$$(t)\mathrm{T} \tag{55}$$

In contrast with *t*, the indefinite object *a* is ambivalent (see explanations above). So we accept the next axiom:

$$(a)\{\mathrm{T, F}\} \tag{56}$$

Axiom (56) is equivalent to: $\{(a)\mathrm{T}\}, \{(a)\mathrm{F}\}$ (see previous paragraph).

$$((A)\{\mathrm{T, F}\})n \tag{57}$$

Axiom (57) is equivalent to: $((A)\mathrm{T})n, ((A)\mathrm{F})n$.

$$((A)a)\mathrm{T} \tag{58}$$

(58) means that any thing has some properties.

$$((a*)A)\mathrm{T} \tag{59}$$

(59) means that any thing belongs to something as a property.

$$(a(A))\mathrm{T} \tag{60}$$

(60) means that any thing has some relations.

$$(A(*a))\mathrm{T} \tag{61}$$

(61) means that any thing is realized in something as a relation.

Note the application of the principle of syntactical priority in the axioms (58–61). If the indefinite object *a* was fixed before entering a syntactical relation, we would have no guarantee that the arbitrary object A has it as a property that (58) states. But if we proceed from a syntactical relation, we can always find objects to satisfy (58).

The next four axioms determine the relations between true objects and false properties, or relations.

$$(([(A)\mathrm{T}])[(a)\mathrm{F}])\mathrm{F} \quad (62)$$

$$(([(A)\mathrm{T}]^*)[(a)\mathrm{F}])\mathrm{F} \quad (63)$$

$$([(a)\mathrm{F}]([(A)\mathrm{T}]))\mathrm{F} \quad (64)$$

$$([(a)\mathrm{F}](^*[(A)\mathrm{T}]))\mathrm{F} \quad (65)$$

The next formulae speak about the falsity of corresponding relations.

$$(\{(A)A\} , [(A)A])\mathrm{F} \quad (66)$$

$$(\{A(A)\} , [A(A)])\mathrm{F} \quad (67)$$

$$(\{(A^*)A\})\mathrm{F} \quad (68)$$

$$(\{A(^*A)\})\mathrm{F} \quad (69)$$

$$([(A^*)A] , [A(^*A)])\mathrm{T} \quad (70)$$

Axiom (70) supposes that there are such properties and relations that belong to all objects irrespective of the "truth" or "falsity" of these objects. E.g., any object has some relations to other objects; any object may be interpreted as a system, etc. While the mentioned objects may be true or false; any property (or relation) that characterize all objects must be true – otherwise this property (or relation) cannot characterize true objects.

$$([(A^*)Lt] , [Lt\,(^*A)])\{\mathrm{T},\ \} \quad (71)$$

$$([(t^*)Lt])\{\mathrm{T},\ \} \quad (72)$$

2.1.4. *Axioms that are implicative relations between elementary WFF that contain no iota operators.*

$$A \supset a \quad (73)$$

$$A \succ a \quad (74)$$

$$([(a)\mathrm{T}] \Rightarrow A)n \quad (75)$$

$$([(a)\mathrm{T}] \succ A)n \quad (76)$$

We must append some comments to these axioms. There is no analog of the axioms (73–74) referring to the attributive implication, because the relation $A \Rightarrow a$ will be proven as a theorem. There is no analog of the axioms (75–76) referring to the mereological implication because the relation $([(a)\mathrm{T}] \supset A)n$ is not valid for the Universum that includes everything in itself.

$$(t)\mathrm{T} \rightarrow (Lt , \overset{\cup}{t})\mathrm{T} \quad (77)$$

$$(a)\{\mathrm{T,F}\} \rightarrow \{ (t', \overset{\Delta}{t}, \overset{\circ}{t})\{\mathrm{T,F}\} \} \quad (78)$$

$$(\{a, t'\} \rightarrow \{ t,\ Lt\ , \overset{\cup}{t}\ \})\{\mathrm{T,F}\} \tag{79}$$

$$a \Rightarrow \{ t \vee t' \} \tag{80}$$

2.1.5. *Axioms that contain iota operators, but no implications*.

$$([([(\iota A)\mathrm{T}]^*)A] , [A(^*[(\iota A)\mathrm{T}])])\mathrm{T} \tag{81}$$

This axiom means that any property or relation of a true thing is itself true.

$$(([([(A)\iota A])\mathrm{T}])\, \iota A)\mathrm{T} \tag{82}$$

This axiom says that an arbitrary thing, actually possessing some arbitrary chosen property, does possess it.

$$((\iota A)[([(\iota A^*)A])\mathrm{T}])\mathrm{T} \tag{83}$$

According to (83), any object possesses its arbitrary true property.

$$([\, \iota A\, \mathrm{K}\, \iota A\,])\{\mathrm{T},\ \} \tag{84}$$

$$([\, \iota A([(\iota A)\mathrm{T}])])\mathrm{T} \tag{85}$$

$$([Lt(t)])\mathrm{T} \tag{86}$$

$$(\iota A\, \mathrm{K}\, \iota A)\mathrm{T} \tag{87}$$

The axioms (84–87) are in need of philosophical comments. They suppose that any thing can be represented as a relation between its elements. The formula $[\ t\ (\ t\ )]$, one of those included in (84) due to the interpretation of K, denotes the thing *t*, which has itself as a relation. And it is a true object according the axiom (55). $[\iota a(\iota a)]$ is an ambivalent object, because a may be false. At the same time the open formula $\iota a(\iota a)$, which is a particular case of (87), is true.

2.1.6. *Axioms that contain both iota operators and implications*.

This group of axioms is the largest one. We will make some stipulations that will help to shorten the formulae records.

In the case when *both attributive and mereological* implications hold between the same things we shall use the symbol $\iota\!\!\rightarrow$ as an abbreviation. Thus, the axiom (73) $A \supset a$, together with the theorem $A \Rightarrow a$, can be written as “one” formula: $A\, \iota\!\!\rightarrow a$. If the sign $\iota\!\!\rightarrow$ is repeated in different places in a formula, this will be interpreted as if in all occurrences are the implication denoted by $\iota\!\!\rightarrow$ is of the same type.

In the case when *all types of implication* (the attributive, mereological, relational and, hence, neutral) hold between the same things, we shall use the symbol $\iota\iota\!\!\rightarrow$ as an abbreviation. If the sign $\iota\iota\!\!\rightarrow$ is repeated in different places in a formula, this will be interpreted as if in all occurrences of the implication denoted by $\iota\iota\!\!\rightarrow$ are of the same type.

If implications hold in the both directions, we shall use the complex signs corresponding to the various types of implications:

$$\Leftrightarrow, \quad \supset\subset, \quad \succ\prec, \quad \leftrightarrow, \quad \iota\!\leftrightarrow, \quad \iota\iota\!\leftrightarrow .$$

$$\{\{(\iota A, \iota\iota A)\mathrm{T}\} \bullet \{\iota\iota A \;\iota\iota\!\!\rightarrow \iota\iota\iota A\}\} \rightarrow \{\iota A, \iota\iota\iota A\} \tag{88}$$

$$\{\{(\iota A \bullet \iota\iota A)\mathrm{T}\} \bullet \{\iota\iota A \;\iota\iota\!\!\rightarrow \iota\iota\iota A\}\} \rightarrow \{\iota A \bullet \iota\iota\iota A\} \tag{89}$$

$$\{\{\iota A \;\iota\iota\!\!\rightarrow \iota\iota A\} \bullet \{\iota A \leftrightarrow \iota\iota\iota A\} \bullet \{\iota\iota A \leftrightarrow \iota\iota\iota\iota A\}\} \rightarrow \{\iota\iota\iota A \;\iota\iota\!\!\rightarrow \iota\iota\iota\iota A\} \tag{90}$$

$$\{\iota A \supset \iota\iota A\} \rightarrow \{\iota A \bullet A \supset \iota\iota A\} \tag{91}$$

$$\{\iota A \supset \{\iota\iota A \bullet A\}\} \rightarrow \{\iota A \supset \iota\iota A\}\} \tag{92}$$

Axioms (91–92) allow us to strengthen the antecedent and to weaken the consequent of mereological implication.

$$\iota A \;\iota\!\!\rightarrow \iota\overset{\cup}{A} \tag{93}$$

$$t \supset Lt \tag{94}$$

$$\iota\overset{\Delta}{A} \;\iota\!\!\rightarrow \iota A \tag{95}$$

$$\{(\iota A)\mathrm{T}, (\iota A)\{\mathrm{T,F}\}\} \Rightarrow \{(\iota A)\{\mathrm{T},\}\} \tag{96}$$

$$\{(\iota A)\mathrm{F}, (\iota A)\{\mathrm{T,F}\}\} \Rightarrow \{(\iota A)\{\mathrm{F},\}\} \tag{97}$$

The next axioms consider various kinds of transitivity:

$$\{\{\iota A \;\iota\iota\!\!\rightarrow \iota\iota A\} \bullet \{\iota\iota A \;\iota\iota\!\!\rightarrow \iota\iota\iota A\}\} \rightarrow \{\iota A \;\iota\iota\!\!\rightarrow \iota\iota\iota A\} \tag{98}$$

$$\{\{\iota A \supset \iota\iota A\} \bullet \{\iota\iota A \Rightarrow \iota\iota\iota A\}\} \rightarrow \{\iota A \supset \iota\iota\iota A\} \tag{99}$$

$$\{\{\iota A \Rightarrow \iota\iota A\} \bullet \{\iota\iota A \supset \iota\iota\iota A\}\} \rightarrow \{\iota A \supset \iota\iota\iota A\} \tag{100}$$

$$\{\{\iota A \succ \iota\iota A\} \bullet \{\iota\iota A \Rightarrow \iota\iota\iota A\}\} \rightarrow \{\iota A \succ \iota\iota\iota A\} \tag{101}$$

$$\{\{\iota A \Rightarrow \iota\iota A\} \bullet \{\iota\iota A \rightarrow \iota\iota\iota A\}\} \rightarrow \{\iota A \rightarrow \iota\iota\iota A\} \tag{102}$$

$$\{\{\iota A \rightarrow \iota\iota A\} \bullet \{\iota\iota A \Rightarrow \iota\iota\iota A\}\} \rightarrow \{\iota A \rightarrow \iota\iota\iota A\} \tag{103}$$

$$\{(\iota A \;\iota\iota\!\!\rightarrow \iota\iota A)\{\mathrm{T},\}\} \bullet \{(\iota\iota A \;\iota\iota\!\!\rightarrow \iota\iota\iota A)\{\mathrm{T},\}\} \rightarrow \{(\iota A \;\iota\iota\!\!\rightarrow \iota\iota\iota A)\{\mathrm{T},\}\} \tag{104}$$

The next axioms consider the interconnections of different types of implication:

$$\{\iota A \Rightarrow \iota\iota A\} \Rightarrow \{\iota A \supset \iota\iota A\} \tag{105}$$

$$\{\iota A \supset \iota\iota A\} \Rightarrow \{\iota A \rightarrow \iota\iota A\} \tag{106}$$

$$\{\iota A \succ \iota\iota A\} \Rightarrow \{\iota A \rightarrow \iota\iota A\} \tag{107}$$

The next axioms express some traditional “Laws of Thought”:

$$\iota A \Leftrightarrow \iota A \tag{108}$$

This is the axiom of identity.

$$((\iota A)\mathrm{T} \bullet (\iota A)\mathrm{F})\mathrm{F} \tag{109}$$

This is the principle of non-contradiction.

$$((\iota A)\mathrm{T} \vee (\iota A)\mathrm{F})\mathrm{T} \qquad (110)$$

This is the principle of excluded middle.

Next are axioms of categorical syllogism:

$$\{\iota A \Rightarrow \iota\iota A\} \to (([(A)\iota A])[([(A)\iota\iota A]^*)A])\mathrm{T} \qquad (111)$$

$$\{\{\iota A \Rightarrow \iota\iota A\} \bullet \{((\iota\iota\iota A)\iota A)\mathrm{T}\}\} \to \{((\iota\iota\iota A)\iota\iota A)\mathrm{T}\} \qquad (112)$$

$$\{(\iota A)[(a^*)\iota\iota A]\} \to \{(\iota A^*)\iota\iota A\} \qquad (113)$$

(113) is the refinement of the famous principle: “Nota notae est nota rei ipsius”.

Next are axioms of disjunctive syllogism:

$$\{\iota A \Rightarrow \{\iota\iota A \vee \iota\iota\iota A\}\} \to \{(\iota A \Rightarrow \iota\iota A)\mathrm{F} \to (\iota A \Rightarrow \iota\iota\iota A)\mathrm{T}\} \qquad (114)$$

$$\{\iota A \Rightarrow \{\iota\iota A \; \mathsf{W} \; \iota\iota\iota A\}\} \to \{(\iota A \Rightarrow \iota\iota A)\mathrm{T} \to (\iota A \Rightarrow \iota\iota\iota A)\mathrm{F}\} \qquad (115)$$

(Note that exclusive disjunction W was defined in part II, analogous to ∨.)

$$\{\{\iota A \vee \iota\iota A\} \bullet \{\{\iota A \; \iota\iota\!\to \iota\iota\iota A\} \bullet \{\iota\iota A \; \iota\iota\!\to \iota\iota\iota A\,)\}\}\} \to \iota\iota\iota A \qquad (116)$$

(116) is the simple constructive dilemma.

$$\{\{\iota A \vee \iota\iota A\} \bullet \{\{\iota A \; \iota\iota\!\to \iota\iota\iota A\} \bullet \{\iota\iota A \; \iota\iota\!\to \iota\iota\iota\iota A\}\}\} \to \{\iota\iota\iota A \vee \iota\iota\iota\iota A\} \qquad (117)$$

(117) is the complex constructive dilemma.

$$\{\{\iota A' \vee \iota\iota A'\} \bullet \{\{\iota\iota\iota A \; \iota\iota\!\to \iota A\} \bullet \{\iota\iota\iota A \; \iota\iota\!\to \iota\iota A\,)\}\} \to \iota\iota\iota A' \qquad (118)$$

(118) is the simple destructive dilemma.

$$\{\{\iota A' \vee \iota\iota\iota A'\} \bullet \{\{\iota\iota\iota A \; \iota\iota\!\to \iota A\} \bullet \{\iota\iota A \; \iota\iota\!\to \iota\iota\iota\iota A\,)\}\}\} \to \{\iota\iota\iota A' \vee \iota\iota A'\} \qquad (119)$$

(119) is the complex destructive dilemma.

$$\{\{\iota A \; \iota\iota\!\to \{\iota\iota A \vee \iota\iota\iota A\}\} \bullet \{\{\iota\iota A \; \iota\iota\!\to \iota\iota\iota\iota A\} \bullet \{\iota\iota\iota A \; \iota\iota\!\to \iota\iota\iota\iota A\}\}\} \to \{\iota A \; \iota\iota\!\to \iota\iota\iota\iota A\} \qquad (120)$$

(120) is a variant of the simple constructive dilemma.

Next are axioms of the direct attributive restriction:

$$\{\iota A \Rightarrow \iota\iota A\} \to \{[(\iota A)\iota\iota\iota A] \Rightarrow [(\iota\iota A)\iota\iota\iota A]\} \qquad (121)$$

The reasoning scheme (121) is known in traditional logic as “The restriction by the third concept”. Its validity depends on the “point” or “non-point” character of a restrictive property (see Uyemov, 1955; Subbotin, 1969; Djdjan, 1977). E.g., in “*A tortoise is an animal. Therefore a rapid tortoise is a rapid animal*” the conclusion is wrong, because *rapid* is a non-point property. It is a linear property – one can be *more* or *less rapid*. In contrary, the inference “*A tortoise is an animal. Therefore a sea tortoise (a turtle) is a sea animal*” is valid, because *sea* is a point property – it is impossible to be more or less *sea*. In (121) the point character of the restrictive property is expressed by the identity of the iota operator $\iota\iota\iota$ applied to it. Contrarily, the inference “*A tortoise is an animal. Therefore a rapid tortoise is a rapid animal*” has the following formal scheme:

$$\{\iota A \Rightarrow \iota\iota A\} \to \{[(\iota A)a] \Rightarrow [(\iota\iota A)a]\} \qquad (122)$$

where a is not necessarily a point property. The occurrence of a in the antecedent of the attributive implication may have another meaning than its occurrence in the consequent. In the particular case of the previous formula we have

$$\{\iota A \Rightarrow \iota\iota A\} \rightarrow \{[(\iota A)[(a)\iota\iota\iota A]] \Rightarrow [(\iota\iota A)\,[(a)\iota\iota\iota A]]\} \qquad (123)$$

Here subformulae $[(a)\iota\iota\iota A]$ do not express obligatory a point property. E.g. "Tomatoes are vegetables. So, tomatoes from *your* (1-st occurrence of a) kitchen-garden ($\iota\iota\iota A$) are vegetables from *my* (2-nd occurrence of a) kitchen-garden ($\iota\iota\iota A$)."

$$\{[(\iota A)\iota\iota\iota A] \Rightarrow [(\iota\iota A)\iota\iota\iota A]\} \rightarrow \{\iota A \Rightarrow \iota\iota A\} \qquad (124)$$

This postulates the validity of the inverse conclusion, e.g. "*A sea tortoise is a sea animal. Therefore a tortoise is an animal*", that may be called a conclusion by rejection.

$$\{\iota A \rightarrow \iota\iota A\} \leftrightarrow \{[(\iota A)\iota\iota\iota A] \rightarrow [(\iota\iota A)\iota\iota\iota A]\} \qquad (125)$$

E.g. "*If the cold weather comes, we shall wear a warm clothes. Therefore, if the cold weather will come tomorrow, we shall wear warm clothes tomorrow*."

$$\{\iota A \Rightarrow \iota\iota A\} \leftrightarrow \{[(\iota\iota\iota A)\iota A] \Rightarrow [(\iota\iota\iota A)\iota\iota A]\} \qquad (126)$$

It is a "restriction *of* (not *by*) the third concept". E.g. "*The square is a rectangle. Therefore this square blackboard is a rectangular blackboard*" and vice versa.

$$\{\iota A \rightarrow \iota\iota A\} \leftrightarrow \{[(\iota\iota\iota A)\iota A] \rightarrow [(\iota\iota\iota A)\iota\iota A]\} \qquad (127)$$

E.g. "*If somebody is in search of something, he will find it. If the good is being searched for by somebody, the good will be found by him*."

Next are axioms of the inverse attributive restriction:

$$\{\iota A \Rightarrow \iota\iota A\} \leftrightarrow \{[(\iota\iota\iota A^*)\iota A] \Rightarrow [(\iota\iota\iota A^*)\iota\iota A]\} \qquad (128)$$
$$\{\iota A \rightarrow \iota\iota A\} \leftrightarrow \{[(\iota\iota\iota A^*)\iota A] \rightarrow [(\iota\iota\iota A^*)\iota\iota A]\} \qquad (129)$$
$$\{\iota A \Rightarrow \iota\iota A\} \leftrightarrow \{[(\iota A^*)\iota\iota\iota A] \Rightarrow [(\iota\iota A^*)\iota\iota\iota A]\} \qquad (130)$$
$$\{\iota A \rightarrow \iota\iota A\} \leftrightarrow \{[(\iota A^*)\iota\iota\iota A] \rightarrow [(\iota\iota A^*)\iota\iota\iota A]\} \qquad (131)$$

Next are axioms of the direct relational restriction:

$$\{\iota A \Rightarrow \iota\iota A\} \leftrightarrow \{[\iota\iota\iota A(\iota A)] \Rightarrow [\iota\iota\iota A(\iota\iota A)]\} \qquad (132)$$
$$\{\iota A \rightarrow \iota\iota A\} \leftrightarrow \{[\iota\iota\iota A(\iota A)] \rightarrow [\iota\iota\iota A(\iota\iota A)]\} \qquad (133)$$
$$\{\iota A \Rightarrow \iota\iota A\} \leftrightarrow \{[\iota A(\iota\iota\iota A)] \Rightarrow [\iota\iota A(\iota\iota\iota A)]\} \qquad (134)$$
$$\{\iota A \rightarrow \iota\iota A\} \leftrightarrow \{[\iota A(\iota\iota\iota A)] \rightarrow [\iota\iota A(\iota\iota\iota A)]\} \qquad (135)$$

Next are axioms of the inverse relational restriction:

$$\{\iota A \Rightarrow \iota\iota A\} \leftrightarrow \{[\iota A(*\iota\iota\iota A)] \Rightarrow [\iota\iota A(*\iota\iota\iota A)]\} \qquad (136)$$
$$\{\iota A \rightarrow \iota\iota A\} \leftrightarrow \{[\iota A(*\iota\iota\iota A)] \rightarrow [\iota\iota A(*\iota\iota\iota A)]\} \qquad (137)$$
$$\{\iota A \Rightarrow \iota\iota A\} \leftrightarrow \{[\iota\iota\iota A(*\iota A)] \Rightarrow [\iota\iota\iota A(*\iota\iota A)]\} \qquad (138)$$
$$\{\iota A \rightarrow \iota\iota A\} \leftrightarrow \{[\iota\iota\iota A(*\iota A)] \rightarrow [\iota\iota\iota A(*\iota\iota A)]\} \qquad (139)$$

Next are axioms of the "reistic" restriction:

$$\{\iota A \;\iota\!\rightarrow\; \iota\iota A\} \;\leftrightarrow\; \{\{\iota A \bullet \iota\iota\iota A\} \;\iota\!\rightarrow\; \{\iota\iota A \bullet \iota\iota\iota A\}\} \tag{140}$$

In the next axioms of this group the metalanguage symbol K, defined earlier, is used.

$$\{[\iota A \,\mathrm{K}\, \iota\iota A] \;\iota\!\rightarrow\; [\iota A \,\mathrm{K}\, \iota\iota\iota A]\} \rightarrow \{\{\{\iota A \bullet \iota\iota\iota\iota A\} \,\mathrm{K}\, \iota\iota A\} \rightarrow \{\{\iota A \bullet \iota\iota\iota\iota A\} \,\mathrm{K}\, \iota\iota\iota A\}\} \tag{141}$$

$$\{[\iota A \,\mathrm{K}\, \iota\iota A] \;\iota\!\rightarrow\; [\iota A \,\mathrm{K}\, \iota\iota\iota A]\} \rightarrow \{\{\iota A \,\mathrm{K}\, \{\iota\iota A \bullet \iota\iota\iota\iota A\}\} \rightarrow \{\iota A \,\mathrm{K}\, \{\iota\iota\iota A \bullet \iota\iota\iota\iota A\}\}\} \tag{142}$$

$$\{[\iota A \,\mathrm{K}\, \iota\iota A] \;\iota\!\rightarrow\; [\iota\iota\iota A \,\mathrm{K}\, \iota\iota A]\} \rightarrow \{\{\{\iota A \bullet \iota\iota\iota A\} \,\mathrm{K}\, \iota\iota A\} \rightarrow \{\{\iota\iota\iota A \bullet \iota\iota\iota A\} \,\mathrm{K}\, \iota\iota A\}\} \tag{143}$$

$$\{[\iota A \,\mathrm{K}\, \iota\iota A] \;\iota\!\rightarrow\; [\iota\iota\iota A \,\mathrm{K}\, \iota\iota A]\} \rightarrow \{\{\iota A \,\mathrm{K}\, \{\iota\iota A \bullet \iota\iota\iota A\}\} \rightarrow \{\iota\iota\iota A \,\mathrm{K}\, \{\iota\iota A \bullet \iota\iota\iota A\}\}\} \tag{144}$$

The next axioms are analogous to (141–144), but contain *closed* formulae *linked by* $\iota\!\rightarrow$ in their consequents:

$$\{[\iota A \,\mathrm{K}\, \iota\iota A] \;\iota\!\rightarrow\; [\iota A \,\mathrm{K}\, \iota\iota\iota A]\} \leftrightarrow \{[\{\iota A \bullet \iota\iota\iota\iota A\} \,\mathrm{K}\, \iota\iota A] \;\iota\!\rightarrow\; [\{\iota A \bullet \iota\iota\iota\iota A\} \,\mathrm{K}\, \iota\iota\iota A]\} \tag{145}$$

$$\{[\iota A \,\mathrm{K}\, \iota\iota A] \;\iota\!\rightarrow\; [\iota A \,\mathrm{K}\, \iota\iota\iota A]\} \leftrightarrow \{[\iota A \,\mathrm{K}\, \{\iota\iota A \bullet \iota\iota\iota\iota A\}] \;\iota\!\rightarrow\; [\iota A \,\mathrm{K}\, \{\iota\iota\iota A \bullet \iota\iota\iota\iota A\}]\} \tag{146}$$

$$\{[\iota A \,\mathrm{K}\, \iota\iota A] \;\iota\!\rightarrow\; [\iota\iota\iota A \,\mathrm{K}\, \iota\iota A]\} \leftrightarrow \{[\{\iota A \bullet \iota\iota\iota A\} \,\mathrm{K}\, \iota\iota A] \;\iota\!\rightarrow\; [\{\iota\iota\iota A \bullet \iota\iota\iota A\} \,\mathrm{K}\, \iota\iota A]\} \tag{147}$$

$$\{[\iota A \,\mathrm{K}\, \iota\iota A] \;\iota\!\rightarrow\; [\iota\iota\iota A \,\mathrm{K}\, \iota\iota A]\} \leftrightarrow \{[\iota A \,\mathrm{K}\, \{\iota\iota A \bullet \iota\iota\iota A\}] \;\iota\!\rightarrow\; [\iota\iota\iota A \,\mathrm{K}\, \{\iota\iota A \bullet \iota\iota\iota A\}]\} \tag{148}$$

The next group of axioms expresses our conception of the interrelations between the categories "Thing", "Property" and "Relation" (Ujomov, 1965). The assertions that a property of a given object can be represented in the form of a relation (*relativization axiom*); and, conversely, a relation can be represented in a form of property (*attributivization axiom*); may be expressed in the following formula:

$$[(a^*)\iota A] \;\Leftrightarrow\; [\iota A(^*a)] \tag{149}$$

The possibility of the interpretation of any property or relation as a thing is expressed by the *substantivation* axioms (150–153):

$$[(A)\iota A] \;\Rightarrow\; \iota A \tag{150}$$

$$[(\iota A^*)\iota A] \;\Leftrightarrow\; \iota A \tag{151}$$

$$[\iota A(\iota A)] \;\Leftrightarrow\; \iota A \tag{152}$$

$$[\iota A(^*\iota A)] \;\Leftrightarrow\; \iota A \tag{153}$$

$$\iota A \;\Rightarrow\; [(\iota A)\iota A] \tag{154}$$

$$[(\iota A^*)A] \;\rightarrow\; \iota A \tag{155}$$

$$[A(^*\iota A)] \;\rightarrow\; \iota A \tag{156}$$

The next axioms postulate the existence of different implications between the results of TDL operations.

$$\{\iota A \bullet \iota A\} \Leftrightarrow \iota A \tag{157}$$

$$\{\iota A \bullet A\} \supset \iota A \tag{158}$$

$$[\iota A(A)] \succ \iota A \tag{159}$$

$$[a(\iota A)] \supset \iota A \tag{160}$$

$$\iota A \ \iota\!\rightarrow [(\iota A^*)A] \tag{161}$$

$$(\iota A)\{\iota\iota A \bullet A\} \Rightarrow (\iota A)\iota\iota A \tag{162}$$

$$\{\{\iota A \mathrm{K} \iota\iota A\}, [\iota A \mathrm{K} \iota\iota A]\} \Rightarrow \{a(^*\{\iota A \bullet \iota\iota A\})\} \tag{163}$$

$$\iota\{\iota A \bullet \iota\iota A\} \Leftrightarrow \{\iota A \supset \iota\iota A\} \tag{164}$$

$$[(\iota A)\mathrm{T}] \Rightarrow \iota A \tag{165}$$

Next are axioms that allow to change the direction of predication:

$$(\iota A)\iota\iota A \leftrightarrow (\iota A^*)\iota\iota A \tag{166}$$

$$\iota\iota A(\iota A) \leftrightarrow \iota\iota A(^*\iota A) \tag{167}$$

$$[(\iota A)\iota\iota A] \leftrightarrow [(\iota A^*)\iota\iota A] \tag{168}$$

$$[\iota\iota A(\iota A)] \leftrightarrow [\iota\iota A(^*\iota A)] \tag{169}$$

The next group of axioms concerns the relations between open and closed formulae of the same “interior”. The truth of a closed formula secures the truth of the open formula corresponding to it but not vice versa. A closed formula is not always true when the corresponding open formula is true. E.g. the open formula $\{(t')a\}$, which states that t' has some properties, is true irrespective of the truth or falsity of t'. But the closed formula $[(t')a]$ is not true, it is ambivalent only due to the ambivalence of t'. Nevertheless, any closed formula is true if its initial place is occupied by a true object and non-initial by an ambivalent one. So we accept the following axioms:

$$\{[(\iota A)\mathrm{T}] \mathrm{K} [(\iota\iota A)\{\mathrm{T,F}\}]\} \rightarrow [\iota A \mathrm{K} \iota\iota A] \tag{170}$$

$$\{\iota A \mathrm{K} \iota\iota A\} \rightarrow ([\iota A \mathrm{K} \iota\iota A])\{\mathrm{T},\}\tag{171}$$

$$[\iota A] \rightarrow \{\iota A\} \tag{172}$$

The next axioms consider the shifting of iota operators:

$$\iota[(A)\iota\iota A] \Leftrightarrow [(\iota A)\iota\iota A] \tag{173}$$

$$\iota[(\iota\iota A^*)A] \Leftrightarrow [(\iota\iota A^*)\iota A] \tag{174}$$

$$\iota[\iota\iota A(A)] \Leftrightarrow [\iota\iota A(\iota A)] \tag{175}$$

$$\iota[A(^*\iota\iota A)] \Leftrightarrow [\iota A(^*\iota\iota A)] \tag{176}$$

The last two groups of axioms postulate some properties for different kinds of implication:

$$\{(\iota A)\iota\iota A\} \rightarrow \{\iota A \Rightarrow [(\iota A^*)\iota\iota A]\} \tag{177}$$

$$\{\iota\iota A(\iota A)\} \rightarrow \{\iota A \succ [\iota\iota A(^*\iota A)]\} \tag{178}$$

$$\{\iota A \Rightarrow \iota\iota A\} \leftrightarrow \{(\iota A)\iota\iota A\} \tag{179}$$

$$\{\iota A \succ \iota\iota A\} \leftrightarrow \{\iota\iota A(\iota A)\} \qquad (180)$$
$$\{\iota A\ \iota\!\!\rightarrow [(\iota\iota A)\{\mathrm{T,F}\}]\} \rightarrow ([(\iota\iota A)\iota A])\{\mathrm{T,}\ \} \qquad (181)$$
$$(\iota A \Rightarrow \iota\iota A)\{\mathrm{T,}\ \} \rightarrow ([(\iota A)\iota\iota A])\{\mathrm{T,}\ \} \qquad (182)$$
$$\{([(\iota A)\iota\iota A])\{\mathrm{T,}\ \}\} \leftrightarrow \{([(\iota\iota A)\iota A])\{\mathrm{T,}\ \}\} \qquad (183)$$
$$[([(\iota\iota A)\iota A])\iota A] \Leftrightarrow [(\iota\iota A)\iota A] \qquad (184)$$
$$\{\{\iota A\ \iota\iota\!\!\rightarrow \iota\iota A\} \bullet \{\iota A\ \iota\iota\!\!\rightarrow \iota\iota\iota A\}\} \leftrightarrow \{\iota A\ \iota\iota\!\!\rightarrow \{\iota A \bullet \iota\iota\iota A\}\} \qquad (185)$$
$$\{\{\iota A \Rightarrow \iota\iota A\} \bullet \{\iota\iota A \Rightarrow \iota\iota\iota A\}\} \rightarrow \{[(\iota A)\iota\iota A] \Rightarrow [(\iota\iota A)\iota\iota\iota A]\} \qquad (186)$$
$$\{\{\iota A \supset \iota\iota A\} \bullet \{\iota\iota A \supset \iota\iota\iota A\}\} \rightarrow \{\{\iota A \bullet \iota\iota A\} \supset \{\iota\iota A \bullet \iota\iota\iota A\}\} \qquad (187)$$
$$\{\{\iota A \succ \iota\iota A\} \bullet \{\iota\iota A \succ \iota\iota\iota A\}\} \rightarrow \{\{\iota A \bullet \iota\iota A\} \succ \{\iota\iota A \bullet \iota\iota\iota A\}\} \qquad (188)$$

2.2. RULES OF INFERENCE

2.2.1. *Terminology and sign conventions*.

We will say that a formula ιA is *derivable* from a formulae list $\iota\iota A$ (list of hypotheses) if ιA is the last formula of a formulae sequence, where any sequence element is either an axiom, or an element of $\iota\iota A$, or a consequence of some preceding elements according to one of the rules of inference listed below. In this case we will use the notation $\iota\iota A \vdash \iota A$.

A formula ιA of the TDL is called *provable* if it is derivable from an empty list of hypotheses. In this case we will use the notation $\vdash \iota A$.

If $\vdash$ is used, and the formula after it ends with the symbol T, this symbol may be omitted, and also T can be ascribed to the end of a formula preceded by $\vdash$.

Let us use the following terminology in the formulations given below. If a formula, which has the valent ending T, is provable, we shall say that its nucleus is *accepted*. If there is a provable formula with the valent ending {T, }, we shall say that its nucleus is *quasi accepted*. If a formula with the valent ending F is provable, we shall say that its nucleus is *contrarily refuted*. Correspondingly, if a formula with the valent ending {F, } is provable, we shall say that its nucleus is *quasi contrarily refuted*.

If a formula with the valent ending T)n or F)n is provable, we shall say that its nucleus – with valent ending T or F – is *contradictory refuted*. (Let us remind the reader that the valent ending T may be omitted in *implications*: e.g. $(A \rightarrow A)n$ is an abbreviation for $((A \rightarrow A)\mathrm{T})n$.) If there is a provable formula with the valent ending {F, }, we shall say that its nucleus is *quasi refuted*.

Unlike axioms labeled by their numbers, there is a tradition of referring to rules of inference by their abbreviations like *MP – modus ponens* in traditional logic. We will follow the tradition that the formulation of a rule will be preceded by its abbreviation typed in italic. If there is a group of kindred rules, their abbreviations will have common beginnings that will appear in the name of paragraph.

2.2.2. *The substitution rules* (*RA*).

RA_0. The acceptability (or quasi acceptability, or contrarily refutation, or quasi contrarily refutation) of a formula is preserved if we substitute the arbitrary occurrence of A in this formula for any formula under the following conditions:

a. The original formula is not closed and the substituted A does not occur in its closed subformula.

b. The substituted A is not bound by a jay or iota operator.

c. The substituted A does not occur in a contradictory refuted subformula of the original formula.

d. If the formula that replaced A contains a single iota operator, then this iota operator must occur somewhere in the original formula.

If a formula or subformula that contains A is closed, the substitutions into A may be still possible under some additional conditions. Let us begin with the simplest case, when there is *only one closed subformula containing A in the original formula, and this closed subformula has only one ending*.

RA_1. If the original formula is closed or A does occur in its closed subformula (the subformula may coincide with the whole formula), then the acceptability (or quasi acceptability, or quasi contrarily refutation) of the original formula is preserved if we substitute this subformula for any formula under the following conditions:

a. A is the *initial* occurrence in the closed subformula.

b. A is not bound by a jay or iota operator.

c. A does not occur in a contradictory refuted subformula of the original formula.

d. If the formula that replaced the subformula contains a single iota operator, then this iota operator must occur somewhere in the original formula.

e. The substituting formula must be closed and have a non-initial ending that coincides with the non-initial part of the original closed subformula.

Note that conditions *b–d* coincide with those of the rule RA_0. The condition *e* means that the substituting formula must fulfill the limitations that are imposed by the non-initial part of the original closed subformula. E.g., if we have a closed formula $[(A)t]$, then we can substitute it for any closed formula that has t as its ending. Thus a substituting formula may be $[(a)t]$ or $[(t)t]$, but not $[(a)\mathring{t}]$.

There can be four types of closed formulae where t occupies the position of formula ending. Excluding the one that was considered above, there are $[t(A)]$, $[(t^*)A]$, $[A(^*t)]$. The examples of valid substitutions into these formulae are respectively: $[t(t')]$, $[(t^*)Lt]$, $[\overset{\Delta}{t}(^*t)]$. The examples of wrong substitutions are: $[\mathring{t}(t')]$, $[(\mathring{t}^*)Lt]$, $[\overset{\Delta}{t}(^*\overset{\cup}{t})]$.

Let us designate a non-initial place of any two-membered closed formula by the symbol e (from "*e*nding"). Let us denote an initial part of a substituting formula by the symbol f. Let the complex sign $[(fA)e]$ denote the result of the substitution of A in the formula $[(A)e]$ for f.

The condition e of the rule RA_1 in the case when RA_1 is applied to the initial occurrence of A in the two-placed closed formula requires the coincidence of the non-initial parts of the substituted and the substituting closed formulae. Formally it can be expressed as a requirement to check the provability of one of the following formulae:

$\vdash$ ([(*A*, *fA*)*e*]){T, } ; $\vdash$ ([(*e**){*A*, *fA*}]){T, } ;
$\vdash$ ([*e*(*A*, *fA*)]){T, } ; $\vdash$ ([{*A*, *fA*}(* *e*)]){T, } .

A simple case of the substitution of [(*A*)*t*] for [(*a*)*t*] was considered above, where the role of *e* was played by *t*.

Let e_1 denotes the ending in a formula of the type [(*A*)*e*]. Analogously let e_2 be an ending in [*e*(*A*)], e_3 – in [(*e**)*A*], e_4 – in [*A*(**e*)]. Thus the index ***i*** in e_i makes it possible to restore the structure of the formula in which e_i appears, so let us agree to use more simple denotations [(*A*) e_1] , [(*A*) e_2] , [(*A*) e_3] , [(*A*) e_4] instead of [(*A*)*e*], [*e*(*A*)], [(*e**)*A*], [*A*(**e*)].

In the more complicated cases a closed formula with *A* in the initial place may have three, four or more non-initial parts (endings). E.g., in the case of two endings there are 16 types of such formulae; some of them are [([(*A*) *e*]) *e*], [*e*([(*A*) *e*])], [([*e*(*A*)]) *e*], [*e*([(*e**)*A*])], [[*A*(**e*)](**e*)] .

Taking into account the previous agreement we can rewrite the above formulae in the next forms: [([(*A*)e_1])e_1], [([(*A*)e_1])e_2], [([(*A*)e_2])e_1], [([(*A*)e_3])e_2], [([(*A*)e_4])e_4]. We can simplify this using the abbreviations: (*A*)11, (*A*)12, (*A*)21, (*A*)32, (*A*)44. Thus the complete list of the 16 cases of two-ending formulae will be: (*A*)11, (*A*)12, (*A*)21, (*A*)22, (*A*)13, (*A*)31, (*A*)23, (*A*)32, (*A*)33, (*A*)14, (*A*)41, (*A*)24, (*A*)42, (*A*)34, (*A*)43, (*A*)44.

If there are three endings, we have 16*4=64 cases. In general, for *n* endings we will have 4^n variants.

RA'_1. The condition *e* of the rule RA'_1 that covers the general case is analogous to the condition *e* of the rule RA_1: it requires the coincidence of the *all non-initial parts* of the substituted and the substituting subformulae.

Let us take the case (*A*)113 as an example. This abbreviation denotes the formula [([([([(*A*)e_1])e_1])e_3]. The rule RA'_1 in this case manifests the demand of the relations:

$\vdash$ ([([([(*A*, *fA*)e_1])e_1])e_3]){T,} .

The concrete example is the deducibility

$\vdash$ ([(*a**)[([(*A*,*Lt*)*t*])*t*]]){T,}

allows the substitution of *A* for *Lt* in [(*a**)[([(*A*)*t*])*t*]] .

Note that if a closed subformula occurs in a list or in an *open* formula, then a substitution of the initial occurrence of *A* in this subformula is realized independently from the part of a formula, which is outer relative to the subformula. If, on the other hand, a list or an open subformula (with the initial occurrence of A) occupies the initial part of a closed formula, then a substitution is admissible only with observance of conditions imposed by another part of that closed formula. E.g., consider the formula $([a([(\{(A)A\})t])])Lt$. Here the open subformula $(A)A$ in braces occupies the initial part of the closed subformula, which imposes that the result of a substitution must have the property t and the relation a. As for subformula Lt, it is out of the considered closed subformula and therefore, it can be neglected when the substitution is executed. E.g. the next substitution in the previous formula is allowed:

$$([a([(a)t])])[a([(t')t])])Lt$$

Here $[(a)t]$ is taken to substitute the first A, and $[(t')t]$ substitutes the second. The conditions imposed by the rule RA'_1 are fulfilled here. Of course, the result of the substitution has different structure than the original formula, but it is inessential because we do not use the rule of coincidence of those structures.

On the contrary, the substitution

$$([t([(a)t])])[a([(t')t])])Lt$$

is not allowed because of the presence of t instead of a in the original formula.

RA''_1. Let the entry of A in a formula be bound by a jay operator. Then this entry of A can be replaced by an arbitrary formula (with preservation of the jay operator) in the case, when the jay operator denotes an object *being identified* (bold-faced letter “j” – see previous Part I). Acceptability and quasi acceptability of the original formula are preserved.

This rule is a consequence of the definition of jay–identity and the rule RA_0. From the definition of jay–identity it follows that the A being identified is a component of an open formula (in the definiens of the definition) and hence RA_0 allows free substitutions in that A. At the same time, if A stands in the position of an object, *to which the identification is made* (i.e. there used italic letter “*j*” in front of this A), it is a component of the non-initial part of a closed subformula, and therefore we have no rights for free substitutions.

RA'''_1. If A occurs in an implication, then any substitutions are *admissible in the antecedent* and *not admissible* if the substituted occurrence is a part of the *consequent*.

This rule is implied by the previous RA''_1. The definitions of all types of implications refer the object being identified to the antecedent, and the object to which the identification is made to the consequent.

The presence of iota operators in the definitions of the implications is not an obstacle for substitutions as we will see in the next rule.

RA_2. If various occurrences of A in a formula are bound *by the same iota operator*, then all these occurrences may be *simultaneously* substituted for *an arbitrary, but the same formula* (preserving this iota operator). If the formula that substitutes A contains the same iota operator that precedes that A, it must be *renamed* within the substituting formula (see rule $RH3$ for renaming below). The following four types of deducibility - provability, quasi provability, contrarily refutation and quasi contrarily refutation – are conserved by substitution of that type.

E.g, in the formula $(\iota A)\iota A$ we may simultaneously substitute both A for a and receive $(\iota a)\iota a$. But if we take not a, but $[(\iota a)a]$ as a substituting firmula, we must rename the iota operator ι (e.g., for $\iota\iota$), and thus obtain $(\iota\,[(\iota\iota a)a])\iota\,[(\iota\iota a)a]$ after the substitution.

$RA3$. Let a formula possess one of the four deducibility types shown in the previous rule, $RA2$. Its deducibility type is preserved in each of the following cases:

a. an occurrence of ιA in the original formula is substituted for ιa or for t.

b. an occurrence of $\iota A'$ is substituted for $\iota a'$ or for t'.

c. an occurrence of $\breve{\iota A}$ is substituted for $\breve{\iota a}$ or for $\breve{t}$.

d. an occurrence of $\overset{\Delta}{\iota A}$ is substituted for $\overset{\Delta}{\iota a}$ or for $\overset{\Delta}{t}$.

e. an occurrence of $\mathring{\iota A}$ is substituted for $\mathring{\iota a}$ or for $\mathring{t}$.

f. an occurrence of $L\iota A$ is substituted for $L\iota a$ or for Lt'.

$RA4$. Contrarial refutation of a formula is preserved if any occurrence of a is substituted for any WFF.

2.2.3. *Introduction rule for the attributive implication* (*RB*).

Let us take a formula of TDL and fix some sequence of its mutually excluded subformulae that fulfills the condition that any symbol of the original formula occurs in some formula of that sequence. Then let us replace each of these subformulae in the original formula by a. We will obtain so-called a-derivative formula. E.g., if we take the formula $((a([(a^*)t]))[a(t)])[([(a)a])a]$, then some of its a-derivative formulae will be: a, $(a)a$, $((a)a)a$, $(a(a))[(a)a]$, $((a(a))a)[(a)a]$, $a([(a)a])$,$((a([(a^*)a]))[a(a)])[([(a)a])a]$.

We assume that any formula attributively implies its a-derivative formulae. If all or some of these formulae will be united into one formula – a free list of a-derivative formulae, - the implication is still valid. This assumptions are fixed by the following rule:

RB. Any formula arttibutively implicates a free list of its a-derivative formulae.

On the base of *RB* we will have, for example:

$$\vdash ((a([(a^*)t]))[a(t)])[([(a)a])a] \Rightarrow \{\, a,\ (a)a,\ ((a)a)a,\ (a(a))[(a)a],$$
$$((a(a))a)[(a)a],\ \ a([(a)a]),\ ((a([(a^*)a]))[a(a)])[([(a)a])a] \,\}$$

2.2.4. *The rules for the lists* (*RC*).

RC_1. If a related list of formulae is accepted then any formula obtained from this list by deleting its arbitrary component is also accepted.

RC_{2a}. If $\iota\iota A$ is a related list of formulae and ιA is a given formula of that list, then the following mereological implication is derivable from $\iota\iota A$:

$$\iota\iota A \ \vdash \ \iota\iota A \supset \iota A\,.$$

RC_{2b}. If $\iota\iota A$ is a free list of formulae and ιA is a given formula of that list, then the following neutral implication is derivable from $\iota\iota A$:

$$\iota\iota A \ \vdash \ \iota\iota A \to \iota A\,.$$

RC_3. If a free list $\{\iota a, \{\iota\iota a, \iota\iota\iota a\}\}$ (or related list $\{\iota a \bullet \{\iota\iota a \bullet \iota\iota\iota a\}\}$) possesses some type of provability or refutation, then the same type is valid for the list $\{\{\iota a, \iota\iota a\}, \iota\iota\iota a\}$ (respectively $\{\{\iota a \bullet \iota\iota a\} \bullet \iota\iota\iota a\}$).

This rule postulates the associativity for lists.

Note that we do not formulate the rule for changing the order of components in the free or related list because such a rule is the simple consequence of the definitions of the free list (see the part I, p. 360) and related list (see the part II, definition 3.2).

RC_4. If the formulae ιA and $\iota\iota A$ both possess the same type of provability or refutation, then this type is also valid for each of the two lists $\{\iota A, \iota\iota A\}$ and $\{\iota A \bullet \iota\iota A\}$.

RC_5. If a component of a free or related list is derivable from other formulae of that list, then this component may be deleted from the list. At the same time, a formula that is derivable from arbitrary formulae of a list may be added to it. In both cases the type of provability or refutation of the list is preserved.

2.2.5. *Introduction rule for* T*-ending* (*RD*).

RD. If the formula ιA possesses some type of provability or refutation, the same type is valid for the formula (ιA^*)T.

2.2.6. *The rules for reduction of valent endings* (*RE*).

RE_1. If the sequence of valent endings, starting with its second element, consists of the sign T only, then the sequence can be truncated on its arbitrary position, starting from the second.

RE_2. Two adjacent signs F in the sequence of valent endings may be substituted for one ending T.

2.2.7. *The rules of definitions* (*RF*).

The first rule in this group concerns definitions having the form “Definitio per Genus proximum et Differentia specifica”, i.e. the definiens has the structure $[(a)a]$, where a in parentheses denotes a thing – Genus proximum – and a out of parentheses denotes a property – Differentia specifica.

RF_1. If Genus proximum does not coincide with A, an attributive implication with the definiendum of a definition as the antecedent and the Genus proximum as the consequent is acceptable. If there is an iota operator ι before the subformula that denotes Genus proximum, and ι does not occur in the definiendum, this iota operator must be omitted according to the rule of such omission given below.

E.g., let us take the definition

$$t' =_{def} \; [(\iota a)\{ (\{ \iota a \supset t \} \cdot \{ t \supset \iota a \})\mathrm{F} \}]$$

(see formula 14). The attributive implication

$$\vdash t' \Rightarrow a$$

is provable based on of RF_1.

RF_2. ***The rule of marriage***. Let an object ιA has two properties specifically – values of two attributive system parameters. This means that this object has a related list of these properties. The definition of that related list supposes:

a. Iota operator ι , put before the object which is characterized by the properties is preserved in the expressions of both properties.

b. The other iota operators must be changed so that there will be no common iota operators in the expressions of both properties.

c. Both components of the related list must be connected by the dot.

E.g., taking the definitions of a structurally non-variable system and an internal-centric system (see part II, 4.8 and this part, formulae 23, 31), due to RF_2 we receive:

$$(\iota A)\text{Structurally non-variable and internal-centric system} =_{def}$$

$$(\iota A)\{ \{\{([\iota\iota a(*\iota A)])t\}\cdot\{[A(*\iota A)]\Rightarrow \iota\iota a\}\}$$

$$\cdot\{\{([a(*\iota A)])t\}\cdot\{ \iota\iota\iota \,\{ \iota\overset{\cup}{A} \}\cdot\{[A(*\iota A)]\Rightarrow[a(*\iota\iota\iota \,\{ \iota\overset{\cup}{A} \})]\}\}\}\}$$

RF_2 may be spread to several members of related list.

RF_3. If some formula is provable then it may be added to any list of formulae that is the definiens of a definition or part of it. Vice versa, a provable formula being a component of a list that is a definiens or part of it, may be deleted from it. In both cases the truth of the definiens is preserved.

*RF*4. The definiendum and the definiens of a definition are both true. The valent ending T can be put or omitted after the definiendum and the definiens. There is only one exception to this rule – the definition of a “true” formula (part II, 3.11).

2.2.8. *The rule for accepting the ambivalence* (*RG*).

RG. If a formula that contains no iota operators is accepted and if a not-initial place of this formula is occupied by some closed subformula, then this closed subformula is ambivalent one.

E.g., if it will be proved that $\vdash((A)[(a)a])\mathrm{T}$, then *RG* gives $\vdash(\,[(a)a]\,)\{\mathrm{T},\mathrm{F}\}$.

2.2.9. *The rules for deletion, raising and renaming of iota operators* (*RH*).

RH_1. If two iota operators ι and $\iota\iota$ are ascribed to the same object within a formula (if we have $\iota\{\iota\iota\iota A\}$ and $\iota\iota\{\iota\iota\iota A\}$ as its subformulae), then in that formula we may replace all iota operators ι by iota operator $\iota\iota$ (and vice versa $\iota\iota$ – by ι). In this case the formula preserves its provability or refutability type.

$RH_{2\text{–}1}$. If a formula that contains occurrences of iota operators is accepted (or contrarily refuted), then so is the formula obtained from the original *by deletion of those iota operators* under the following conditions:

a. In scopes of application of iota operators there present *only definite, indefinite or iotified objects* (i.e. there are *no free*, not bounded by another iota or jay operators, *occurrences of arbitrary object A*).

b. The deletion of iota operators before subformulae containing the arbitrary objects *A* is still possible if:

1. these arbitrary objects *occupy such places where substitutions are forbidden*, or
2. after the deletion of iota operators before subformulae containing *A there appears only* ***one*** *free for substitutions occurrence of the arbitrary object A in a formula*. It concerns either arbitrary (RA_0) or restrictive (RA_1) substitutions.

c. It is assumed that the result of the deletion *should not contain any single iota operator*.

$RH_{2\text{–}2}$. ***The rule of divorce***. If two parts of a formula are connected with iota operator before entries of *A* and this iota operator is the only one in both parts of the formula, it may be deleted in the case when both parts of the formula are considered as separate formulae, i.e. as members of a free list (e.g. as a result of applying the rules below RC_1 or *MP*, separation of definiens from definiendum, etc.)

This rule is a conclusion from the $RH_{2\text{–}1}$. The occurrence of A may be such that either any substitution in it is forbidden or free or restricted substitutions in it are allowed. In the first case we can delete the iota operator on the basis of $RH_{2\text{–}1}b_1$. In the second case – on the basis of $RH_{2\text{–}1}b_2$, because after the division of the original formula there will be only one entry of A which will be free for substitutions. Acceptability or quasi acceptability of the original formula transforms into acceptability or quasi acceptability of both formulae of the received free list.

RH_3. Every iota operator may be renamed if its new name does not currently occur in the formula. This renaming does not influence the type of provability or refutability of the formula.

Except where iota operators must be renamed, avoiding formulae unwieldiness (see rule RA_2 above) is recommended. For this purpose it may be convenient also to use number indexes like ι_1 , ι_2 , instead of repeating the letter ι .

RH_4. The acceptability of an attributive implication is preserved if the iota operator, which does not occur in this implication, is ascribed both to its antecedent and to its consequent.

RH_5. Synonymy preserves itself if we add or delete the same iota operator before the left and the right parts connected by $=_{syn}$.

RH_6. If a formula is given in which

- *a*. an iota operator binds two or several *identical* subformulae;
- *b*. the subformulae mentioned in item *a* include some subformula, the initial places of which are not occupied by A;
- *c*. the subformula mentioned in item *b* is not directly preceded by any iota operator;

then in all identical subformulae mentioned in *a*, before the occurrence of the subformula, mentioned in *b,c*, it is possible to *add a new iota operator* currently absent in the original formula. The type of provability or refutability of the original formula is preserved in such a case.

E.g., from $\{\iota[(t')a],\ \iota[(t')a]\}$ it is possible to deduce (applying RH_6 twice) the formula $\{\iota[(\iota_2 t')\iota_3 a],\ \iota[(\iota_2 t')\iota_3 a]\}$. Here it is necessary to take into account that the definition of t' (Part II, formula 3.39) does not contain A in its initial place.

RH_7. If a formula contains two or more related lists with the same number of components and the same iota operators that bind all corresponding members of those lists (e.g. $\{\iota A \bullet \iota\iota\, a \bullet \iota\iota\iota\, t'\}$ and $\{\iota\, a \bullet \iota\iota A \bullet \iota\iota\iota a\}$), then the acceptability of that formula is preserved if all mentioned iota operators will be deleted and a new iota operator will be placed in front of all corresponding lists, on the condition that this new iota operator was not present in the formula.

Thus the lists $\{\iota A \bullet \iota\iota a \bullet \iota\iota\iota\, t'\}$ and $\{\iota a \bullet \iota\iota A \bullet \iota\iota\iota a\}$ can be transformed to $\iota_4\{A \bullet a \bullet t'\}$ and $\iota_4\{a \bullet A \bullet a\}$.

RH_8. If a formula includes a free or related list of the identical subformulae, the provability of it is preserved if we put an iota operator, currently not present in the formula, before each member of the list.

E.g., $(A \bullet A \bullet A)a$ can be proved, so due to RH_8 $(\iota A \bullet \iota A \bullet \iota A)a$ will be provable, too.

RH_9. Two or more entries of A in places which permit free substitutions may be united with the common iota operator. Acceptability and quasi acceptability of a formula are preserved in this case.

Really, if we can substitute A for any formulae, we can take two formulae that begin with the same iota operator. E.g., in the formula $(A)A$ we can substitute both occurrences of A for ιA and receive $(\iota A)\iota A$.

RH_{10}. If the entries of A occur in the initial places of closed subformulae of a formula, and the rule RA_1 allows restrictive substitutions in these occurrences of A, the common iota operator can be put in front of these occurrences only in the case when *non-initial parts of the closed subformulae are identical*. Acceptability and quasi acceptability of the original formula are preserved.

This rule is a consequence of the rule RA_1 and is analogous to RH_9.

2.2.10. *The rules for implications usage* (*RI*).

There are two general conditions that must be fulfilled for the valid usage of *any* rule in this paragraph, therefore, we formulate these conditions before the formulations of rules:

a. If a formula obtained as a result of a rule application contains an iota operator unpaired within that formula, this iota operator must be erased using the above rules RH_{2-1} or RH_{2-2}.

b. A contextual indefiniteness, not bound by a iota operator, will not become an initial in the result of a rule application (if it happens, the rule is not applicable).

RI_1. ***Modus ponens***: If we accept an implication of any kind, and accept also its antecedent, then we must accept the consequent of this implication.

We will use for *Modus Ponens* its usual abbreviation *MP* together with RI_1.

RI'_1. If the implication of any kind is *accepted* or *quasi accepted*, and the formula that indicates the quasitruth of its antecedent is provable, then the formula that indicates the quasitruth of the consequent is provable.

E.g., if it is proven: 1) $\vdash (a \Rightarrow \mathring{t})\{\mathrm{T},\ \}$ and 2) $\vdash (a)\{\mathrm{T},\ \}$, then 3) $\vdash (\mathring{t})\{\mathrm{T},\ \}$ will be stated.

RI_2. ***Modus tollens***: If an implication of any kind is accepted, but its consequent is contrarily or contradictorily refuted, then we must refute (resp. contrarily or contradictorily) its antecedent.

We will use for *Modus Tollens* its usual abbreviation *MT* together with RI_2.

RI'_2. If the implication of any kind is *accepted* or *quasi accepted*, and the formula that indicates the quasifalsity of its consequent is provable, then the formula that indicates the quasifalsity of the antecedent is provable.

E.g., if it is proven: 1) $\vdash (a \Rightarrow \mathring{t})\{\mathrm{T},\}$ and 2) $\vdash (\mathring{t})\{\mathrm{F},\}$, then 3) $\vdash (a)\{\mathrm{F},\}$ will be stated.

Let us reproduce schemes of definitions for the contradictory negations given in the Part II (the numeration of these schemes in Part II is preserved):

$$((\iota\iota A\ \mathrm{Q}\ \{A\ \mathrm{R}\ \iota A\})\mathrm{T})n =_{def} (\iota\iota A\ \mathrm{Q}\ \{a\ \mathrm{R}\ \iota A\})\mathrm{F} \qquad \text{(Part II, 3.26)}$$

$$((\iota\iota A\ \mathrm{Q}\ \{A\ \mathrm{R}\ \iota A\})\mathrm{F})n =_{def} (\iota\iota A\ \mathrm{Q}\ \{a\ \mathrm{R}\ \iota A\})\mathrm{T} \qquad \text{(Part II, 3.27)}$$

$$((\iota\iota A\ \mathrm{Q}\ \{a\ \mathrm{R}\ \iota A\})\mathrm{T})n =_{def} (\iota\iota A\ \mathrm{Q}\ \{A\ \mathrm{R}\ \iota A\})\mathrm{F} \qquad \text{(Part II, 3.34)}$$

$$((\iota\iota A\ \mathrm{Q}\ \{a\ \mathrm{R}\ \iota A\})\mathrm{F})n =_{def} (\iota\iota A\ \mathrm{Q}\ \{A\ \mathrm{R}\ \iota A\})\mathrm{T} \qquad \text{(Part II, 3.35)}$$

The next rule can be derived from these schemes:

RI_3. Let the antecedent of the accepted implication coincide with the nucleus of the definiendum (resp. definiens) of the given scheme chosen from the above four, and its consequent coincides with the nucleus of the corresponding definiens (resp. definiendum). These conditions are satisfied by the following implication: $\{\iota\iota A\,\mathrm{Q}\,\{A\,\mathrm{R}\,\iota A\}\} \to \{\iota\iota A\,\mathrm{Q}\,\{a\,\mathrm{R}\,\iota A\}\}$. According to the definitions reproduced above there are four cases:

1) If we contradictorily refute the truth of the antecedent, then we must contrarily refute the consequent.

2) If we contradictorily refute the falsity of the antecedent, then we must accept the consequent.

3) If we contrarily refute the antecedent, then we must contradictorily refute the truth of the consequent.

4) If we accept the antecedent, then we must contradictorily refute the falsity of the consequent.

E.g., let us take $A \Rightarrow t$ for the antecedent of the discussed implication, and $a \Rightarrow t$ for its consequent (see Part II, p.141). We will have:

$$((A \Rightarrow t)\mathrm{T})n \text{ means } (a \Rightarrow t)\mathrm{F};$$

$$((A \Rightarrow t)\mathrm{F})n \text{ means } (a \Rightarrow t)\mathrm{T};$$

$$(A \Rightarrow t)\mathrm{F} \text{ means } ((a \Rightarrow t)\mathrm{T})n;$$

$$(A \Rightarrow t)\mathrm{T} \text{ means } ((a \Rightarrow t)\mathrm{F})n.$$

2.2.11. *The replacement rules* (*RJ*).

*RJ*1. Let 1) an attributive equivalence ($\Leftrightarrow$), or 2) synonymy ($=_{syn}$), or 3) the definiendum or the definiens of some definition ($=_{def}$), be accepted. Let a given formula *includes* one (left or right) part of (1), or (2), or (3). The type of provability or refutability of this formula is preserved *if one part of an equivalence*, *synonymy*, or *definition* in it *is replaced by the other part*, provided that there are no occurrences of the same iota operators in the replaced parts and the rest of the formula. If present, such operators must be renamed or deleted on the basis of the rule RH_{2-1}. The rule is valid if there is no change of iota operators as a result of the transformation, i.e. if a subformula with an iota operator is changed for a formula with the same iota operator.

*RJ*2. Let an implication of any kind with *open* antecedent and *open* consequent have a definite type of provability or refutability. This type will be preserved if we replace these open antecedent and consequent by the corresponding *closed* formulae. Vice versa, if both antecedent and consequent in the implication are closed, they may be replaced by corresponding open formulae.

E.g., if we have: $(A)a\rightarrow(t)a$, we have also: $[(A)a]\rightarrow[(t)a]$.

*RJ*3. The type of provability or refutability of a formula which includes subformula of the type $A\,\mathrm{R}\,A$ (see the definition of R in part II, formula 3.21) is preserved, if in place of the property or relation we substitute *open* subformula for the corresponding *closed* one, or vice versa.

E.g., if we have the provability: $t\rightarrow(t)\{(t)a\}$, we have also: $t\rightarrow(t)[(t)a]$.

*RJ*4. Acceptability of a formula is preserved if a non-initial subformula of that formula is replaced by a.

E.g., if we accept $(t)[(a)t]$, we have also $(t)a$.

2.2.12. *Leonenko's rule* (*RK*).

RK. Let some formula ιA contains a subformula $\iota\iota A$. If ιA is accepted, then the formula $(\iota\iota A)\iota A$ is also accepted:

$$(\iota A)\mathrm{T} \vdash ((\iota\iota A)\iota A)\mathrm{T} .$$

E.g. Consider the axiom (179):

$\vdash \{\iota A \Rightarrow \iota\iota A\} \leftrightarrow \{(\iota A)\iota\iota A\}$

The rule RH_3 allows us to rename the iota operator ι and get:

$\vdash \{\iota\iota\iota A \Rightarrow \iota\iota A\} \leftrightarrow \{(\iota\iota\iota A)\iota\iota A\}$

Due to the convention postulated in paragraph 2.2.1, we can describe the symbol T as the ending of this implication:

$\vdash (\{\iota\iota\iota A \Rightarrow \iota\iota A\} \leftrightarrow \{(\iota\iota\iota A)\iota\iota A\})\mathrm{T}$

Now let us use *RK*, taking the last formula ιA and $\iota\iota A$ as its subformula. We obtain:
$(\{\{\iota\iota A \Rightarrow \iota A\} \leftrightarrow \{(\iota\iota A)\iota A\}\})\mathrm{T} \vdash ((\iota A)\{\{\iota\iota A \Rightarrow \iota A\} \leftrightarrow \{(\iota\iota A)\iota A\}\})\mathrm{T}$
The left formula is accepted, so we get
$\vdash ((\iota A)\{\iota\iota A \Rightarrow \iota A\} \leftrightarrow \{(\iota\iota A)\iota A\})\mathrm{T}$.

2.2.13. *The rule for adding and deleting of curly brackets* (*RL*).

RL. Curly brackets can be used in order to avoid the ambiguity that may arise during the formulae transformation. If there is no ambiguity, curly brackets may be omitted. In particular, curly brackets are added if a formula becomes a subformula and are omitted if a subformula closed in them becomes a formula in its own right.

E.g., the expression $(A)a(A)$ is ambiguous, because it can be understand as either "*A* has the property $a(A)$" or "*A* has the relation $(A)a$" . So we must use $(A)\{a(A)\}$ or $\{(A)a\}(A)$ instead.

Because of the simplicity of this rule, we will often use it without the explicit references.

2.2.14. *The rules for closed formulae* (*RM*).

RM_1. Any closed formula, in which the initial places are occupied by the arbitrary objects (A), and the non-initial places are occupied by true or ambivalent subformulae, is quasitrue.

E.g., from $\vdash (t)\mathrm{T}$ we deduce $\vdash ([(A)t])\{\mathrm{T},\ \}$; analogously from $\vdash (a)\{\mathrm{T}, \mathrm{F}\}$ we can get $\vdash ([(A)a])\{\mathrm{T},\ \}$.

RM_2. The type of provability or refutability of a closed formula, the non-initial place of which is occupied by a, is identical to the type of provability (resp. refutability) of the initial part of that closed formula.

E.g., $(t')\{\mathrm{T,F}\} \vdash ([(t')a])\{\mathrm{T,F}\}$; $(a,t',\mathring{t})\{\mathrm{F},\ \} \vdash ([(a,t',\mathring{t})a])\{\mathrm{F},\ \}$; $(t)\mathrm{T} \vdash ([(a^*)t\ ])\mathrm{T}$; $(t')\{\mathrm{T,F}\} \vdash ([([(t')a])t])\{\mathrm{T,F}\}$.

RM_3. True remains true if it is ascribed to any object as its property or relation.

This rule covers the cases when the object, obtained as a result of the ascribing, is non-existing itself. E.g., the round square has a real property "round" in spite of the fact that the round square does not exist.

E.g., if we have $(t)\mathrm{T}$, then we have also $(a)\{(t)\mathrm{T}\}$. If we have $[(a)\mathrm{T}]$, then we have also $[(a)\mathrm{T}](t)$.

RM_4. Any property or relation of any true object is true.

E.g., if we have $(t)\mathrm{T}$, then we have also $([(t^*)A])\mathrm{T}$. If we have $[(\iota A)\mathrm{T}]$, then we have also $[A(*\iota A)])\mathrm{T}$.

3. SOME THEOREMS OF THE TERNARY DESCRIPTION LANGUAGE

3.1. *Implications of elementary formulae.*

The elementary formulae, in the genuine sense of this word, are only three: t, a, A. As for the derivative objects, like t', $\overset{\cup}{t}$, $\overset{\Delta}{t}$, $\overset{\circ}{t}$, Lt, the definienses of their definitions are rather complicated formulae. But we can consider all of them as an elementary formulae of a sort. All basic and derivative objects are related to one another by implications. We will state these implications as the theorems of the TDL.

$$\vdash A \Rightarrow a \tag{189}$$

To prove it use the introduction rule for the attributive implication RB. The a-derivative formula for the formula A is a. According to RB we obtain (189).

$$\vdash \{ t, Lt, a, t', \overset{\cup}{t}, \overset{\Delta}{t}, \overset{\circ}{t} \} \Rightarrow a \tag{190}$$

The theorem is proven with the help of substitution into the antecedent of the previous theorem according to the rule RA'''_1.

$$\vdash t \,\iota\!\rightarrow \overset{\cup}{t} \tag{191}$$

Take the axiom (93). On the basis of the substitution rule RA_3 substitute ιA for t and $\iota \overset{\cup}{A}$ for $\overset{\cup}{t}$. Receive what must be proven.

$$\vdash \overset{\cup}{t} \Rightarrow t' \tag{192}$$

To obtain (192), take the definition rule RF_1, and apply it to the definition of $\overset{\cup}{t}$.

$$\vdash t \Rightarrow t' \tag{193}$$

The result is received by application of the axiom of transitivity (98) (with the attributive implication $\Rightarrow$ taken for $\iota\iota\!\rightarrow$); the rule RA_2; the rule of the deletion of iota operators $RH_{2\text{–}1}$ (item a); and the theorems (191) – (192). Then we must use rule RC_4 for lists and MP.

$$\vdash \{ \{ t \Rightarrow t \}, \{ Lt \Rightarrow Lt \}, \{ a \Rightarrow a \}, \{ t' \Rightarrow t' \}, \{ \overset{\cup}{t} \Rightarrow \overset{\cup}{t} \}, \{ \overset{\Delta}{t} \Rightarrow \overset{\Delta}{t} \}, \{ \overset{\circ}{t} \Rightarrow \overset{\circ}{t} \} \} \tag{194}$$

Take the law of identity (axiom 108: $\iota A \Rightarrow \iota A$). On the base of the rule RA_2 make the substitutions of both occurrences of A for t , Lt , etc. We can check that in each of the obtained cases the conditions of the rule $RH_{2\text{–}1}$ are satisfied so that the iota operator can be deleted.

$$\vdash a \Rightarrow t' \tag{195}$$

Take the axiom (120). Using RA_2, make the following substitutions: ιA for ιa, $\iota\iota A$ for $\iota\iota t$, $\iota\iota\iota A$ for $\iota\iota\iota t'$, and $\iota\iota\iota\iota A$ for $\iota\iota\iota\iota t'$. Choose the attributive implication $\Rightarrow$ as the meaning of $\iota\iota\!\rightarrow$; then delete iota operators using RH_{2-1} (item a). We will receive:

$$\{\{\ a \Rightarrow \{t \vee t'\}\} \bullet \{\{t \Rightarrow t'\} \bullet \{t' \Rightarrow t'\}\}\ \} \rightarrow \{\ a \Rightarrow t'\}$$

The all three components of the antecedent of the neutral implications are provable. The first is the axiom (80), the second is the theorem (193), the third is implied by the theorem (194). The rule RC_4 states that the whole antecedent is provable. Then the consequent is proved due to MP.

$$\vdash\ A \Rightarrow\ t' \qquad (196)$$

Put together the theorems (189) and (195). The usage of the transitivity axiom (98) and the rules RC_4 and MP gives the possibility to prove (196). It is also necessary to delete iota operators that occur in (98), so we must use $RH_{2-1}a$ and $RH_{2-1}b$ (the last allows to delete ι in front of A).

$$\vdash\ \{\ Lt\ , t', \overset{\Delta}{t}\ ,\ \overset{\circ}{t}\ \} \Rightarrow\ t' \qquad (197)$$

This theorem is proved by substitution into the antecedent of the previous theorem (196) according to the rule RA'''_1.

$$\vdash\ \overset{\Delta}{t}\ \iota\!\rightarrow\ t \qquad (198)$$

Take the axiom (95). Substitute $\iota\overset{\Delta}{A}$ for $\iota\,\overset{\Delta}{t}$, and ιA for ιt on the base of RA_3 and then delete iota operators using $RH_{2-1}a$. We obtain (198).

$$\vdash\ \overset{\Delta}{t}\ \iota\!\rightarrow\ \overset{\cup}{t} \qquad (199)$$

This can be proved by application of the transitivity axiom (98) to the theorems (198) and (191). Use RA_2 and $RH_{2-1}a$, and then the rules RC_4 and MP.

$$\vdash\ Lt \Rightarrow\ t \qquad (200)$$

This can be obtained by using the definition rule RF_1 for the definition of Lt (see Part II, formula 3.43).

$$\vdash\ Lt\ \iota\!\rightarrow\ \overset{\cup}{t} \qquad (201)$$

Put together the previous theorem and the theorem (191). Use the transitivity axioms (98) and (100). Using RA_2 make the substitutions: ιA for ιLt, $\iota\iota A$ for $\iota\iota t$, and $\iota\iota\iota A$ for $\iota\iota\iota\overset{\cup}{t}$. Delete iota operators ($RH_{2-1}a$). We receive from (98), taken $\Rightarrow$ as $\iota\iota\!\rightarrow$:

$$\vdash\ \{\{\ Lt \Rightarrow t\ \} \bullet \{\ t \Rightarrow \overset{\cup}{t}\ \}\} \rightarrow \{\ Lt \Rightarrow \overset{\cup}{t}\ \}$$

Analogously from (100) we receive

$$\vdash\ \{\{\ Lt \Rightarrow t\ \} \bullet \{\ t \supset \overset{\cup}{t}\ \}\} \rightarrow \{\ Lt \supset \overset{\cup}{t}\ \}$$

Put together the above two results, we obtain

$\vdash \{\{ Lt \Rightarrow t \} \bullet \{ t \; \iota\!\!\rightarrow \overset{\cup}{t} \}\} \rightarrow \{ Lt \; \iota\!\!\rightarrow \overset{\cup}{t} \}$

Applying RC_4 and *MP*, we obtain (201) from the last formula.

$$\vdash \overset{\Delta}{t} \supset Lt \tag{202}$$

Accordingly to (198), taking $\supset$ as $\iota\!\!\rightarrow$, we have $\vdash \overset{\Delta}{t} \supset t$. Take the axiom (94): $\vdash t \supset Lt$. Use the transitivity axiom (98). Make the necessary substitutions (RA_2) and delete iota operators ($RH_{2\text{–}1}a$). We will obtain $\vdash \{\{ \overset{\Delta}{t} \supset t \} \bullet \{ t \supset Lt \}\} \rightarrow \{ \overset{\Delta}{t} \supset Lt \}$. Applying RC_4 and *MP*, we finally obtain (202).

3.2. *Theorems that characterize relations between implications.*

$$\vdash \{ \iota A \Rightarrow \iota\iota A \} \Rightarrow \{ \iota A \rightarrow \iota\iota A \} \tag{203}$$

Designate implications $\{ \iota A \Rightarrow \iota\iota A\}$, $\{ \iota A \supset \iota\iota A\}$ and $\{ \iota A \rightarrow \iota\iota A\}$ by the symbols $\iota_3 A$, $\iota_4 A$ and $\iota_5 A$ respectively. Accordingly to the axioms (105) and (106) we obtain: $\vdash \{\iota_3 A \Rightarrow \iota_4 A\}$ and $\vdash \{\iota_4 A \Rightarrow \iota_5 A\}$. Both implications are accepted. Therefore, according to the rule RC_4, the related list $\{\iota_3 A \Rightarrow \iota_4 A\} \bullet \{\iota_4 A \Rightarrow \iota_5 A\}$ is accepted, too. According to the transitivity axiom (98) we obtain

$\vdash \{\{\iota_3 A \Rightarrow \iota_4 A\} \bullet \{\iota_4 A \Rightarrow \iota_5 A\}\} \rightarrow \{\iota_3 A \Rightarrow \iota_5 A\}$

Finally, using *MP*, we obtain $\vdash \{\iota_3 A \Rightarrow \iota_5 A\}$, that is (203).

$$\vdash \{ \iota A \Rightarrow \iota\iota A \} \rightarrow \{ \iota A \rightarrow \iota\iota A \} \tag{204}$$

Let us designate the implication $\{\iota A \Rightarrow \iota\iota A\}$ by the symbol $\iota_3 A$, and the implication $\{\iota A \rightarrow \iota\iota A\}$ by the symbol $\iota_4 A$. Accordingly to (203), we have $\vdash \{ \iota_3 A \Rightarrow \iota_4 A\} \Rightarrow \{ \iota_3 A \rightarrow \iota_4 A\}$, i.e.

$\vdash \{ \{ \iota A \Rightarrow \iota\iota A \} \Rightarrow \{ \iota A \rightarrow \iota\iota A \} \} \Rightarrow \{ \{ \iota A \Rightarrow \iota\iota A \} \rightarrow \{ \iota A \rightarrow \iota\iota A \} \}$

The antecedent of the main implication here is (203), so it is accepted. Therefore, the consequent, i.e. (204), is proved due to *MP*.

$$\vdash \{ \iota A \supset \iota\iota A \} \rightarrow \{ \iota A \rightarrow \iota\iota A \} \tag{205}$$

This theorem can be proven on the basis of the axiom (106) analogous to the proof of (204).

$$\vdash \{ \iota A \succ \iota\iota A \} \rightarrow \{ \iota A \rightarrow \iota\iota A \} \tag{206}$$

This theorem can be proven on the base of the axiom (107) analogous to the proof of (204).

3.3. *Theorems of substantivation.*

$$\vdash \ [(\iota A)\iota A] \Leftrightarrow \iota A \qquad (207)$$

Take the axiom (150): $\vdash [(A)\iota A] \Rightarrow \iota A$. Substitute $[(\iota A)\iota A]$ onto the antecedent accordingly to the rule RA_1. The comparison of the axioms (150) and (2.1.5.4) makes evident the identity of endings (condition e of the rule RA_1). Obtain: $\vdash [(\iota A)\iota A] \Rightarrow \iota A$. Combination of the last result with axiom (154) gives the required theorem.

$$\vdash \ [(\iota A)\ a] \Leftrightarrow \iota A \qquad (208)$$

Take the theorem (207). According to the rule RJ_4 we can substitute the second occurrence of ιA in (207) for a. Thus (208) is accepted.

$$\vdash \ [(a)\{\ a, t, Lt, t', \overset{\cup}{t}, \overset{\Delta}{t}, \overset{\circ}{t}\}] \Leftrightarrow \{\ a, t, Lt, t', \overset{\cup}{t}, \overset{\Delta}{t}, \overset{\circ}{t}\} \qquad (209)$$

Take the axiom (150): $\vdash [(A)\iota A] \Rightarrow \iota A$. Substitute ιA for $\iota\{\ a, t, Lt, t', \overset{\cup}{t}, \overset{\Delta}{t}, \overset{\circ}{t}\}$ on the basis of RA_2 to obtain

$$\vdash \ [(A)\iota\{\ a, t, Lt, t', \overset{\cup}{t}, \overset{\Delta}{t}, \overset{\circ}{t}\}] \Rightarrow \iota\{\ a, t, Lt, t', \overset{\cup}{t}, \overset{\Delta}{t}, \overset{\circ}{t}\}$$

Delete the iota operators on the base of $RH_{2\text{–}1}a$, then substitute the antecedent for $[(a)\iota\{\ a, t, Lt, t', \overset{\cup}{t}, \overset{\Delta}{t}, \overset{\circ}{t}\}]$ on the base of RA_1. The requirement e of RA_1 is fulfilled because of the identity of the lists on the non-initial places. The other requirements of RA_1 are fulfilled as well. We obtain:

$$\vdash \ [(a)\{\ a, t, Lt, t', \overset{\cup}{t}, \overset{\Delta}{t}, \overset{\circ}{t}\}] \Rightarrow \{\ a, t, Lt, t', \overset{\cup}{t}, \overset{\Delta}{t}, \overset{\circ}{t}\}$$

Further, use the axiom (154): $\vdash \iota A \Rightarrow [(\iota A)\iota A]$. Substitute the second ιA for a on the basis of RJ_4: $\vdash \iota A \Rightarrow [(a)\iota A]$. Substitute ιA for $\iota\{\ a, t, Lt, t', \overset{\cup}{t}, \overset{\Delta}{t}, \overset{\circ}{t}\}$ on the basis of RA_2. Then delete the iota operators on the basis of $RH_{2\text{–}1}a$, we obtain

$$\vdash \ \{\ a, t, Lt, t', \overset{\cup}{t}, \overset{\Delta}{t}, \overset{\circ}{t}\} \Rightarrow [(a)\{\ a, t, Lt, t', \overset{\cup}{t}, \overset{\Delta}{t}, \overset{\circ}{t}\}]$$

Put together the two obtained results, we obtain (209).

$$\vdash \ \iota A \Leftrightarrow [(a^*)\iota A] \qquad (210)$$

Use the substantivation axiom (151): $\vdash [(\iota A^*)\iota A] \Leftrightarrow \iota A$. The rule RJ_4 allows us to substitute the first (from the left side) occurrence of ιA for a, so (210) is provable.

$$\vdash \ \iota A \Leftrightarrow [\iota A(^*a)] \qquad (211)$$

This theorem can be proven analogously to the previous one using the axiom (153) instead of (151).

4. THE EXAMPLES OF SYSTEM-THEORETICAL LAWS AS THEOREMS OF T.D.L.

4.1. ***Theorem* 1**. Any structurally open system is a structural-non-point system.

The proof. Let us take the definition of the superobject $\overset{\Delta}{\iota A}$ (see formula 16). Definiendum is defined with the help of $\iota A'$, which is the Genus proximum of the definiens. On the basis of the rule RF_1 we have

$$\vdash \overset{\Delta}{\iota A} \Rightarrow \iota A' \tag{212}$$

From this according to the rule of substitution RA_3 we obtain

$$\vdash \overset{\Delta}{\iota a} \Rightarrow \iota a' \tag{213}$$

Let us take the axiom of the inverse relational restriction (136) and apply the rule of renaming iota operators RH_3 to it:

$$\vdash \{\iota_4 A \Rightarrow \iota_5 A\} \leftrightarrow \{[\iota_4 A(*\iota_6 A)] \Rightarrow [\iota_5 A(*\iota_6 A)]\} \tag{214}$$

Make the following substitutions in (214) using the rule RA_2: substitute $\iota_4 A$ for $\iota_4 \{\overset{\Delta}{\iota a}\}$; $\iota_5 A$ – for $\iota_5 \{\iota a'\}$; $\iota_6 A$ – for $\iota_6 \{\iota\iota\iota A\}$. Obtain:

$$\vdash \{\iota_4 \{\overset{\Delta}{\iota a}\} \Rightarrow \iota_5 \{\iota a'\}\} \leftrightarrow \{[\iota_4 \{\overset{\Delta}{\iota a}\} (*\iota_6 \{\iota\iota\iota A\})] \Rightarrow [\iota_5 \{\iota a'\} (*\iota_6 \{\iota\iota\iota A\})]\} \tag{215}$$

Use the rule of iota operators deletion $RH_{2-1}a$ to delete ι_4, ι_5 and ι_6. Obtain:

$$\vdash \{\overset{\Delta}{\iota a} \Rightarrow \iota a'\} \leftrightarrow \{[\overset{\Delta}{\iota a} (*\iota\iota\iota A)] \Rightarrow [\iota a'(*\iota\iota\iota A)]\} \tag{216}$$

Applying MP to (213) and (216) gives

$$\vdash [\overset{\Delta}{\iota a} (*\iota\iota\iota A)] \Rightarrow [\iota a'(*\iota\iota\iota A)] \tag{217}$$

Here we add and delete curly brackets according to the rule RL.

Now use the axiom of the direct attributive restriction (121) and rename iota operators in it (RH_3). Obtain:

$$\vdash \{\iota_4 A \Rightarrow \iota_5 A\} \rightarrow \{[(\iota_4 A)\iota_6 A] \Rightarrow [(\iota_5 A)\ \iota_6 A]\} \tag{218}$$

Make the following substitutions in (218) using the rule RA_2: substitute $\iota_4\ A$ for $\iota_4 [\overset{\Delta}{\iota a}(*\iota\iota\iota A)]$; $\iota_5 A$ – for $\iota_5 [\iota a'(*\iota\iota\iota A)]$; $\iota_6 A$ – for $\iota_6\, t$. Obtain:

$$\vdash \{\iota_4 [\overset{\Delta}{\iota a}(*\iota\iota\iota A)] \Rightarrow \iota_5 [\iota a'(*\iota\iota\iota A)]\} \rightarrow$$

$$\rightarrow \{[(\iota_4 [\overset{\Delta}{\iota a}(*\iota\iota\iota A)])\iota_6\, t] \Rightarrow [(\iota_5 [\iota a'(*\iota\iota\iota A)])\iota_6\, t]\} \tag{219}$$

Delete iota operators ι_4, ι_5 and ι_6 on the base of $RH_{2-1}a$; obtain

$$\vdash \{[\overset{\Delta}{\iota a}(*\iota\iota\iota A)] \Rightarrow [\iota a'(*\iota\iota\iota A)]\} \rightarrow \{[([\overset{\Delta}{\iota a}(*\iota\iota\iota A)])\, t] \Rightarrow [([\iota a'(*\iota\iota\iota A)])\, t]\} \tag{220}$$

Applying MP to (217) and (220) gives

$$\vdash [([\overset{\Delta}{\iota a}(*\iota\iota\iota A)])\, t] \Rightarrow [([\iota a'(*\iota\iota\iota A)])\, t] \tag{221}$$

Here we have deleted external curly brackets according to the rule *RL*.

Now let us use the rule RJ_2 that allows us to change the closed formulae in the antecedent and consequent by the open ones:

$$\vdash \{([\iota\overset{\Delta}{a}(*\iota\iota A)])\,t\} \Rightarrow \{([\iota a'(*\iota\iota A)])\,t\} \tag{222}$$

Using RH_3, rename iota operator ι as $\iota\iota$, and then rename $\iota\iota$ as ι. Obtain:

$$\vdash \{([\iota\iota\overset{\Delta}{a}(*\iota A)])\,t\} \Rightarrow \{([\iota\iota a'(*\iota A)])\,t\} \tag{223}$$

Now use the axiom of reistic restriction (140), adding to both sides of (223) the definiens of the systems concept definition: $([\iota\iota a(*\iota A)])\,t$. After applying *MP* to thus obtained formula and (223), we obtain:

$$\vdash \{\{([\iota\iota a(*\iota A)])t\}\bullet\{([\iota\iota\overset{\Delta}{a}(*\iota A)])\,t\}\} \Rightarrow \{\{([\iota\iota a(*\iota A)])t\}\bullet\{([\iota\iota a'(*\iota A)])\,t\}\} \tag{224}$$

Now take the axiom (126) and rename iota operators in it:

$$\vdash \{\iota_4 A \Rightarrow \iota_5 A\} \leftrightarrow \{[(\iota_6 A)\iota_4 A] \Rightarrow [(\iota_6 A)\iota_5 A]\} \tag{225}$$

Make the following substitutions in (225) using the rule RA_2:

$\iota_4 A$ – for $\iota_4\{\{([\iota\iota a(*\iota A)])t\}\bullet\{([\iota\iota\overset{\Delta}{a}(*\iota A)])\,t\}\}$;

$\iota_5 A$ – for $\iota_5\{\{([\iota\iota a(*\iota A)])t\}\bullet\{([\iota\iota a'(*\iota A)])\,t\}\}$;

$\iota_6 A$ – for $\iota_6\{\iota A\}$.

Obtain:

$$\vdash \underset{3}{\{}\iota_4 \underset{2}{\{}\underset{1}{\{}([\iota\iota a(*\iota A)])t\underset{1}{\}}\bullet\underset{1}{\{}([\iota\iota\overset{\Delta}{a}(*\iota A)])\,t\underset{1}{\}}\underset{2}{\}} \Rightarrow \iota_5 \underset{2}{\{}\underset{1}{\{}([\iota\iota a(*\iota A)])t\underset{1}{\}}\bullet\underset{1}{\{}([\iota\iota a'(*\iota A)])\,t\underset{1}{\}}\underset{2}{\}}\underset{3}{\}} \leftrightarrow$$

$$\leftrightarrow \underset{3}{\{}[(\iota_6\{\iota A\})\iota_4 \underset{2}{\{}\underset{1}{\{}([\iota\iota a(*\iota A)])t\underset{1}{\}}\bullet\underset{1}{\{}([\iota\iota\overset{\Delta}{a}(*\iota A)])\,t\underset{1}{\}}\underset{2}{\}}] \Rightarrow$$

$$\Rightarrow [(\iota_6\{\iota A\})\iota_5 \underset{2}{\{}\underset{1}{\{}([\iota\iota a(*\iota A)])t\underset{1}{\}}\bullet\underset{1}{\{}([\iota\iota a'(*\iota A)])\,t\underset{1}{\}}\underset{2}{\}}]\underset{3}{\}} \tag{226}$$

In order to avoid confusion we have indexed the braces by numbers here.

Now delete iota operators ι_4, ι_5 and ι_6 in (226) on the basis of $RH_{2-1}a$; then apply *MP* to (224) and the obtained formula. We will obtain the attributive implication ($\Rightarrow$) between closed formulae. Use the rule RJ_2 and replace these closed formulae by open ones. Obtain:

$$\vdash (\iota A)\{\ \{([\iota\iota a(*\iota A)])t\}\bullet\{([\iota\iota\overset{\Delta}{a}(*\iota A)])\,t\}\ \} \Rightarrow$$

$$\Rightarrow (\iota A)\{\ \{([\iota\iota a(*\iota A)])t\}\bullet\{([\iota\iota a'(*\iota A)])\,t\}\ \} \tag{227}$$

The antecedent of (227) coincides with the definiens of the definition of the structurally open system, and the consequent coincides with the definiens of the structural-non-point system. Thus the theorem is proven.

4.2. ***Theorem* 2**. Any gomeomeric system is a non-minimal system.

The proof. Take the axiom of the direct attributive restriction (121) and substitute $\iota\iota A$ for $\iota\iota a$ on the base of RA_2. Receive:

$$\vdash \{\iota A \Rightarrow \iota\iota a\} \rightarrow \{[(\iota A)\iota\iota\iota A] \Rightarrow [(\iota\iota a)\iota\iota\iota A]\} \tag{228}$$

Omit the iota operator $\iota\iota$ before a on the basis of $RH_{2-1}a$. Now we may delete the iota operator ι before A on the base of $RH_{2-1}b$, and receive

$$\vdash \{A \Rightarrow a\} \rightarrow \{[(A)\iota\iota\iota A] \Rightarrow [(a)\iota\iota\iota A]\} \tag{229}$$

because after this deletion substitutions in the second occurrence of non-iotified A will be forbidden since this occurrence is included to the consequent of the implication (see the rule RA'''_1). Therefore, the first occurrence of A remains a single occurrence that is free for substitutions, so $RH_{2-1}b$ is applicable.

The antecedent of (229) is the theorem (189). Therefore, due to MP:

$$\vdash [(A)\iota\iota\iota A] \Rightarrow [(a)\iota\iota\iota A] \tag{230}$$

(the braces were deleted on the basis of RL.) Now apply RA_2 to substitute $\iota\iota\iota A$ for $\iota\iota\iota\overset{\cup}{A}$:

$$\vdash [(A)\iota\iota\iota\overset{\cup}{A}] \Rightarrow [(a)\iota\iota\iota\overset{\cup}{A}] \tag{231}$$

Renaming the iota operator $\iota\iota\iota$ (RH_3) gives:

$$\vdash [(A)\iota\overset{\cup}{A}] \Rightarrow [(a)\iota\overset{\cup}{A}] \tag{232}$$

Now take the axiom of the inverse relational restriction (138) and rename iota operators in it (RH_3) to get:

$$\vdash \{\iota_4 A \Rightarrow \iota_5 A\} \leftrightarrow \{[\iota_6 A(*\iota_4 A)] \Rightarrow [\iota_6 A(*\iota_5 A)]\} \tag{233}$$

Make the following substitutions in (233) using the rule RA_2: substitute $\iota_4 A$ for $\iota_4\,[(A)\iota\overset{\cup}{A}]$; $\iota_5 A -$ for $\iota_5\,[(a)\iota\overset{\cup}{A}]$; $\iota_6 A -$ for $\iota_6\,\{\iota\iota\iota\, A\}$. Obtain:

$$\vdash \{\iota_4\,[(A)\iota\overset{\cup}{A}] \Rightarrow \iota_5\,[(a)\iota\overset{\cup}{A}]\} \leftrightarrow$$

$$\leftrightarrow \{[\iota_6\,\{\iota\iota\iota\, A\}(*\iota_4\,[(A)\iota\overset{\cup}{A}])] \Rightarrow [\iota_6\,\{\iota\iota\iota\, A\}(*\iota_5\,[(a)\iota\overset{\cup}{A}])]\} \tag{234}$$

We can delete the iota operator ι_4 on the basis of $RH_{2-1}b$ because substitutions in A in the second occurrence of $\iota_4[(A)\iota\overset{\cup}{A}]$ are forbidden (it occupies the non-initial place in the closed subformula $[\iota_6\{\iota\iota\iota\, A\}(*\iota_4[(A)\iota\overset{\cup}{A}])]$). Hence, A in the first occurrence of $\iota_4[(A)\iota\overset{\cup}{A}]$ is a single occurrence that is free for substitutions, so $RH_{2-1}b$ is applicable. Furthermore, iota operators ι_5 and ι_6 can be deleted on the base of $RH_{2-1}a$. Thus we obtain:

$$\vdash \{[(A)\iota\overset{\cup}{A}] \Rightarrow [(a)\iota\overset{\cup}{A}]\} \leftrightarrow \{[\iota\iota\iota\, A(*[(A)\iota\overset{\cup}{A}])] \Rightarrow [\iota\iota\iota\, A(*[(a)\iota\overset{\cup}{A}])]\} \tag{235}$$

Now take into account that (232) is the antecedent of (235), and use MP. Deleting the external braces, we obtain:

$$\vdash [\iota\iota\iota\, A(*[(A)\iota\overset{\cup}{A}])] \Rightarrow [\iota\iota\iota\, A(*[(a)\iota\overset{\cup}{A}])] \tag{236}$$

Use the rule RA_2 to substitute $\iota\iota\iota A$ for $\iota\iota\iota a$, and then rename iota operator $\iota\iota\iota$ for $\iota\iota$. We obtain:

$$\vdash [\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])] \Rightarrow [\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])] \tag{237}$$

Now take the axiom of the direct attributive restriction (121) and rename iota operators in it (RH_3) to obtain:

$$\vdash \{\iota_4 A \Rightarrow \iota_5 A\} \rightarrow \{[(\iota_4 A)\iota_6 A] \Rightarrow [(\iota_5 A)\iota_6 A]\} \qquad (238)$$

Using the rule RA_2, substitute in (238): $\iota_4\, A$ for $\iota_4\, [\iota\iota\; a(*[(A)\iota\overset{\cup}{A}\,])]$; $\iota_5\, A$ – for $\iota_5\,[\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])]$; $\iota_6 A$ – for $\iota_6\, t$. Obtain:

$$\vdash \{\iota_4\,[\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])] \Rightarrow \iota_5\,[\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])]\} \rightarrow$$

$$\rightarrow \{[(\iota_4\,[\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])])\iota_6\, t] \Rightarrow [(\iota_5\,[\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\iota_6\, t]\} \qquad (239)$$

After the deletion of iota operators ι_4, ι_5 and ι_6 in (239), possible due to $RH_{2\text{–}1}a$ and $RH_{2\text{–}1}b$ we obtain:

$$\vdash \{\,[\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])] \Rightarrow [\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])]\,\} \rightarrow$$

$$\rightarrow \{[(\,[\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])])\, t] \Rightarrow [(\,[\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\, t]\} \qquad (240)$$

Now apply MP to (237) and (240). Deleting the external curly brackets (RL) we obtain:

$$\vdash [(\,[\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])])\, t] \Rightarrow [(\,[\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\, t] \qquad (241)$$

Take now the axiom of the reistic restriction (140); use $\Rightarrow$ as $\iota\!\rightarrow$; and rename iota operators in it (RH_3) to obtain:

$$\vdash \{\iota_4 A \Rightarrow \iota_5 A\} \leftrightarrow \{\{\iota_4 A \bullet \iota_6 A\} \Rightarrow \{\iota_5 A \bullet \iota_6 A\}\} \qquad (242)$$

Make the following substitutions in (242) using the rule RA_2:

$\iota_4 A$ – for $\iota_4\,\{(\,[\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])])\, t\}$;

$\iota_5 A$ – for $\iota_5\,\{(\,[\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\, t\}$;

$\iota_6 A$ – for $\iota_6\,\{([\iota\iota a(*\iota A)\,])t\}$.

Obtain:

$$\vdash \{\iota_4\,\{(\,[\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])])\, t\} \Rightarrow \iota_5\,\{(\,[\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\, t\}\,\} \leftrightarrow$$

$$\leftrightarrow \{\{\iota_4\,\{(\,[\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])])\, t\} \bullet \iota_6\,\{([\iota\iota a(*\iota A)\,])t\}\} \Rightarrow$$

$$\Rightarrow \{\iota_5\,\{(\,[\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\, t\} \bullet \iota_6\,\{([\iota\iota a(*\iota A)\,])t\}\}\,\} \qquad (243)$$

We can delete iota operator ι_4 because the occurrence of A in its scope occupies the non-initial place of the closed formula, so $RH_{2\text{–}1}b$ is applicable. We can also delete ι_5 and ι_6 on the basis of $RH_{2\text{–}1}a$. Obtain:

$$\vdash \{\,\{(\,[\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])])\, t]\} \Rightarrow \{(\,[\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\, t\}\,\} \leftrightarrow$$

$$\leftrightarrow \{\,\{\{(\,[\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])])\, t\} \bullet \{([\iota\iota a(*\iota A)\,])t\}\} \Rightarrow$$

$$\Rightarrow \{\{([\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\, t\} \bullet \{([\iota\iota a(*\iota A)\ ])t\}\}\ \} \tag{244}$$

Now open the antecedent and the consequent of (241) (rule RJ_2), and apply MP to the result obtained and (244). Then change the order of the list's components. We obtain:

$$\vdash \{\ \{([\iota\iota a(*\iota A)\ ])t\} \bullet \{(\ [\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])])\, t\}\ \} \Rightarrow$$

$$\Rightarrow \{\ \{([\iota\iota a(*\iota A)\ ])t\} \bullet \{(\ [\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\, t\}\ \} \tag{245}$$

Let us now take the axiom of the direct attributive restriction (126) and rename iota operators in it (RH_3) to get:

$$\vdash \{\iota_4 A \Rightarrow \iota_5 A\ \} \leftrightarrow \{[(\iota_6 A)\iota_4 A] \Rightarrow [(\iota_6 A)\iota_5 A]\} \tag{246}$$

Make the following substitutions in (246) using the rule RA_2

$\iota_4 A$ – for $\iota_4\{\ \{([\iota\iota a(*\iota A)\ ])t\} \bullet \{(\ [\iota\iota\, a(*[(A)\iota\overset{\cup}{A}])])\, t\}\ \}$;

$\iota_5 A$ – for $\iota_5\{\ \{([\iota\iota a(*\iota A)\ ])t\} \bullet \{(\ [\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\, t\}\ \}$;

$\iota_6 A$ – for $\iota_6\{\iota A\ \}$.

Obtain:

$$\vdash \underset{3}{\{}\ \iota_4 \underset{2}{\{}\ \underset{1}{\{}\ (\ \underset{1}{[}\iota\iota a(*\iota A)\underset{1}{]}\ )t\underset{1}{\}} \bullet \underset{1}{\{}\ (\ \underset{2}{[}\iota\iota\, a(*\underset{1}{[}(A)\iota\overset{\cup}{A}\underset{1}{]}\ )\ \underset{2}{]}\ )\ t\ \underset{1}{\}}\underset{2}{\}} \Rightarrow$$

$$\Rightarrow \iota_5 \underset{2}{\{}\underset{1}{\{}(\ \underset{1}{[}\iota\iota a(*\iota A)\underset{1}{]}\ )t\underset{1}{\}} \bullet \underset{1}{\{}\ (\ \underset{2}{[}\iota\iota\, a(*\underset{1}{[}(a)\iota\overset{\cup}{A}\underset{1}{]}\ )\ \underset{2}{]}\ )\ t\ \underset{1}{\}}\underset{2}{\}}\ \underset{3}{\}} \leftrightarrow$$

$$\leftrightarrow \underset{3}{\{}\ \underset{4}{[}(\iota_6 \underset{1}{\{}\iota A\underset{1}{\}})\iota_4 \underset{2}{\{}\underset{1}{\{}(\underset{1}{[}\iota\iota a(*\iota A)\underset{1}{]})t\underset{1}{\}} \bullet \underset{1}{\{}(\underset{2}{[}\iota\iota\, a(*\underset{1}{[}(A)\iota\overset{\cup}{A}\underset{1}{]})\underset{2}{]})\ t\underset{1}{\}}\ \underset{2}{\}}\ \underset{4}{]}\ \Rightarrow$$

$$\Rightarrow \underset{4}{[}(\iota_6 \underset{1}{\{}\iota A\underset{1}{\}})\iota_5 \underset{2}{\{}\underset{1}{\{}(\underset{1}{[}\iota\iota a(*\iota A)\underset{1}{]})t\underset{1}{\}} \bullet \underset{1}{\{}(\underset{2}{[}\iota\iota\, a(*\underset{1}{[}(a)\iota\overset{\cup}{A}\underset{1}{]})\underset{2}{]})\ t\underset{1}{\}}\underset{2}{\}}\ \underset{4}{]}\ \underset{3}{\}} \tag{247}$$

Here we have indexed the square and the curly brackets to make the formula more readable.

The rules $RH_{2-1}a$ and $RH_{2-1}b_1$ allow us to delete iota operators ι_4, ι_5 and ι_6. We obtain:

$$\vdash \{\ \{\{([\iota\iota a(*\iota A)\ ])t\} \bullet \{([\iota\iota\, a(*[(A)\iota\overset{\cup}{A}\ ])])\ t\}\ \} \Rightarrow$$

$$\Rightarrow \{\{([\iota\iota a(*\iota A)])t\} \bullet \{([\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\ t\}\}\ \} \leftrightarrow$$

$$\leftrightarrow \{\ [(\iota A)\{\{([\iota\iota a(*\iota A)\ ])t\} \bullet \{([\iota\iota\, a(*[\ (A)\iota\overset{\cup}{A}])])\ t\}\}] \Rightarrow$$

$$\Rightarrow [(\iota A)\{\{([\iota\iota a(*\iota A)\ ])t\} \bullet \{([\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\ t\}\}\ ]\ \} \tag{248}$$

In (248) we omit the indexes of brackets, but the reader can refer to the previous formula (247) if necessary.

Now apply MP to formulae (245) and (248). We obtain:

$$\vdash [(\iota A)\{\{([\iota\iota a(*\iota A)\ ])t\} \bullet \{([\iota\iota\, a(*[\ (A)\iota\overset{\cup}{A}])])\ t\}\}] \Rightarrow$$

$$\Rightarrow [(\iota A)\{\{([\iota\iota a(*\iota A)\ ])t\} \bullet \{([\iota\iota\, a(*[(a)\iota\overset{\cup}{A}])])\ t\}\}\ ] \tag{249}$$

Here we have omitted the external braces on the basis of *RL*.

To continue the proof of Theorem 2 we will prove the following lemma now:

Lemma 1. $\vdash \overset{\cup}{\iota A} \Leftrightarrow [(a)\overset{\cup}{\iota A}]$.

The proof of lemma 1. Take the substantivation axiom (150): $\vdash [(A)\iota A] \Rightarrow \iota A$. Use the substitution rule RA_2 and substitute ιA for $\overset{\cup}{\iota A}$. Obtain:

$$\vdash [(A)\overset{\cup}{\iota A}] \Rightarrow \overset{\cup}{\iota A} \tag{250}$$

Use now the rule RA_1 and substitute $[(A)\overset{\cup}{\iota A}]$, where A is on the initial place in the antecedent of (250), for $[(a)\overset{\cup}{\iota A}]$. Obtain:

$$\vdash [(a)\overset{\cup}{\iota A}] \Rightarrow \overset{\cup}{\iota A} \tag{251}$$

Take now the axiom (154): $\vdash \iota A \Rightarrow [(\iota A)\iota A]$. Substitute ιA for $\overset{\cup}{\iota A}$ (RA_2):

$$\vdash \overset{\cup}{\iota A} \Rightarrow [(\overset{\cup}{\iota A})\overset{\cup}{\iota A}] \tag{252}$$

Use the rule RJ_4 to obtain:

$$\vdash \overset{\cup}{\iota A} \Rightarrow [(a)\overset{\cup}{\iota A}] \tag{253}$$

Putting together (251) and (253) we finish the proof of Lemma 1.

We now return to the proof of Theorem 2. Take the above result (249) and substitute the occurrence of $[(a)\overset{\cup}{\iota A}]$ in its consequent for $\overset{\cup}{\iota A}$. It is possible on the basis of Lemma 1 and the rule RJ_1. The application of RJ_1 is correct, because $[(a)\overset{\cup}{\iota A}]$ and $\overset{\cup}{\iota A}$ have the same iota operator. Obtain:

$$\vdash [(\iota A)\{\{([\iota\iota a(*\iota A)])t\} \bullet \{([\iota\iota\, a(*[(A)\overset{\cup}{\iota A}])])\, t\}\}] \Rightarrow$$

$$\Rightarrow [(\iota A)\{\{([\iota\iota a(*\iota A)])t\} \bullet \{([\iota\iota\, a(*\overset{\cup}{\iota A})])\, t\}\}] \tag{254}$$

In (254) the antecedent coincides with the definiens of gomeomery system, and the consequent coincides with the definiens of non-minimal system. Thus Theorem 2 is proven.

4.3. ***Theorem 3***. Any external-centric system is a nonimmanent system.

The proof. Take the axiom (162), rename the iota operators (RH_3) to receive:

$$\vdash (\iota_4 A)\{\iota_5 A \bullet A\} \Rightarrow (\iota_4 A)\iota_5 A \tag{255}$$

Use the substitution rule RA_2 and substitute in (255) $\iota_4 A$ for $\iota_4\{\iota A\}$ and $\iota_5 A$ for $\iota_5\{([a(*\iota A \bullet \mathring{\iota A})])t\}$. Receive:

$$\vdash (\iota_4\{\iota A\})\{\iota_5\{([a(*\iota A \bullet \mathring{\iota A})])t\} \bullet A\} \Rightarrow (\iota_4\{\iota A\})\iota_5\{([a(*\iota A \bullet \mathring{\iota A})])t\} \tag{256}$$

Omit the iota operators ι_4 and ι_5 on the basis of $RH_{2-1}a$. Obtain:

$$\vdash (\iota A)\{\{([a(*\iota A \bullet \iota \overset{\circ}{A})])t\} \bullet A\} \Rightarrow (\iota A)\{([a(*\iota A \bullet \iota \overset{\circ}{A})])t\} \tag{257}$$

Use now the rule RA_0 and substitute A in the antecedent for the following formula: $\{\iota\iota\,\{\iota\overset{\circ}{A}\} \bullet \{[A(*\iota A)] \Rightarrow [a(*\iota\iota\,\{\iota\overset{\circ}{A}\})]\}\}$. Obtain:

$$\vdash (\iota A)\{\{([a(*\iota A \bullet \iota \overset{\circ}{A})])t\} \bullet \{\iota\iota\{\iota\overset{\circ}{A}\} \bullet \{[A(*\iota A)] \Rightarrow [a(*\iota\iota\{\iota\overset{\circ}{A}\})]\}\}\} \Rightarrow$$
$$\Rightarrow (\iota A)\{([a(*\iota A \bullet \iota \overset{\circ}{A})])t\} \tag{258}$$

In (258) the antecedent coincides with the definiens of external-centric system, and the consequent coincides with the definiens of nonimmanent system. Thus Theorem 3 is proven.

Theorem* 3*a. If a system is external-centric, it is a centric one.

The proof. Look at the definition of external-centric system. Use the rule RJ_4, changing $\iota\overset{\circ}{A}$ by a. Receive the definition of the centric system.

Theorem* 3*b. If a system is internal-centric, it is a centric one.

The proof is analogous to the proof of the previous theorem. Consider $\iota\overset{\cup}{A}$ instead of $\iota\overset{\circ}{A}$.

4.4. ***Theorem* 4**. If a system is non-minimal, it is also non-elementary.

The proof. Compare the definienses of two definitions – (1) non-minimal system:

$$(\iota A)\{\{([\iota\iota\, a(*\iota A)])t\} \bullet \{([\iota\iota\, a(*\iota\overset{\cup}{A})])t\}\}$$

and 2) non-elementary system:

$$(\iota A)\{\{([a(*\iota A)])t\} \bullet \{([a(*\iota\overset{\cup}{A})])t\}\}$$

The difference is only in the presence of iota operators $\iota\iota$ in the first case and their absence in the second. But according to the rule RH_{2-1} (item *a*) we can delete these iota operators in the first definition and thus obtain the second as its consequence. The theorem is proven.

4.5. ***Theorem* 5**. Any non-minimal system is a non-unique one.

The proof. Let us take the definition (15) of the object $\iota\overset{\cup}{A}$:

$$\iota\overset{\cup}{A} =_{def} [(\iota A')\{\iota A \supset \iota A'\}] \tag{259}$$

According to the rule RF_1 this definition implies that the following formula is accepted:

$$\vdash \iota\overset{\cup}{A} \Rightarrow \iota A' \tag{260}$$

Use the axiom of inverse relational restriction (138), rename iota operators in it (RH_3) and obtain:

$$\vdash \{\iota_4 A \Rightarrow \iota_5 A\} \leftrightarrow \{[\iota_6 A(*\iota_4 A)] \Rightarrow [\iota_6 A(*\iota_5 A)]\} \tag{261}$$

Use the substitution rule RA_2 and substitute in (261) $\iota_4 A$ for $\iota_4\{\overset{\cup}{\iota A}\}$, $\iota_5 A$ for $\iota_5\{\iota A'\}$ and $\iota_6 A$ for $\iota_6\{\iota\iota\, a\}$. Obtain:

$$\vdash \{\iota_4\{\overset{\cup}{\iota A}\} \Rightarrow \iota_5\{\iota A'\}\} \leftrightarrow \{[\iota_6\{\iota\iota a\}(*\iota_4\{\overset{\cup}{\iota A}\})] \Rightarrow [\iota_6\{\iota\iota a\}(*\iota_5\{\iota A'\})]\} \qquad (262)$$

Delete iota operators ι_4, ι_5 and ι_6 on the basis of $RH_{2-1}a$, then omit braces on the basis of *RL*

$$\vdash \{\overset{\cup}{\iota A} \Rightarrow \iota A'\} \leftrightarrow \{[\iota\iota a(*\overset{\cup}{\iota A})] \Rightarrow [\iota\iota a(*\iota A')]\} \qquad (263)$$

Apply *MP* to (260) and (263):

$$\vdash [\iota\iota a(*\overset{\cup}{\iota A})] \Rightarrow [\iota\iota a(*\iota A')] \qquad (264)$$

Use the axiom of direct attributive restriction (121) and rename iota operators in it to obtain:

$$\vdash \{\iota_4 A \Rightarrow \iota_5 A\} \rightarrow \{[(\iota_4 A)\iota_6 A] \Rightarrow [(\iota_5 A)\iota_6 A]\} \qquad (265)$$

Make the following substitutions (RA_2): $\iota_4 A$ for $\iota_4[\iota\iota a(*\overset{\cup}{\iota A})]$, $\iota_5 A$ for $\iota_5[\iota\iota a(*\iota A')]$ and $\iota_6 A$ for $\iota_6 t$. Obtain:

$$\vdash \{\iota_4[\iota\iota a(*\overset{\cup}{\iota A})] \Rightarrow \iota_5[\iota\iota a(*\iota A')]\} \rightarrow$$

$$\rightarrow \{[(\iota_4[\iota\iota a(*\overset{\cup}{\iota A})])\iota_6 t] \Rightarrow [(\iota_5[\iota\iota a(*\iota A')])\iota_6 t]\} \qquad (266)$$

Delete iota operators ι_4, ι_5 and ι_6 on the basis of $RH_{2-1}a$, receive:

$$\vdash \{[\iota\iota a(*\overset{\cup}{\iota A})] \Rightarrow [\iota\iota a(*\iota A')]\} \rightarrow \{[([\iota\iota a(*\overset{\cup}{\iota A})])\, t] \Rightarrow [([\iota\iota a(*\iota A')])\, t]\} \qquad (267)$$

Apply *MP* to (264) and (267):

$$\vdash [([\iota\iota a(*\overset{\cup}{\iota A})])\, t] \Rightarrow [([\iota\iota a(*\iota A')])\, t] \qquad (268)$$

Now use the axiom of the reistic restriction (140), taking $\Rightarrow$ as $\iota\!\rightarrow$. Rename iota operators in it to obtain:

$$\vdash \{\iota_4 A \Rightarrow \iota_5 A\} \leftrightarrow \{\{\iota_4 A \bullet \iota_6 A\} \Rightarrow \{\iota_5 A \bullet \iota_6 A\}\} \qquad (269)$$

Make the following substitutions (RA_2): $\iota_4 A$ for $\iota_4[([\iota\iota a(*\overset{\cup}{\iota A})])t]$, $\iota_5 A$ for $\iota_5[([\iota\iota a(*\iota A')])t]$ and $\iota_6 A$ for $\iota_6\{([\iota\iota a(*\iota A)])t\}$. Then change the order of related list components and obtain:

$$\vdash \{\iota_4[([\iota\iota a(*\overset{\cup}{\iota A})])\, t] \Rightarrow \iota_5[([\iota\iota a(*\iota A')])\, t]\} \leftrightarrow$$

$$\leftrightarrow \{\{\iota_6\{([\iota\iota a(*\iota A)])\, t\} \bullet \iota_4[([\iota\iota a(*\overset{\cup}{\iota A})])\, t]\} \Rightarrow$$

$$\Rightarrow \{\iota_6\{([\iota\iota a(*\iota A)])\, t\} \bullet \iota_5[([\iota\iota a(*\iota A')])\, t]\}\} \qquad (270)$$

Delete iota operators ι_4, ι_5 and ι_6 on the basis of $RH_{2-1}a$, then apply *MP* to the result and formula (268); obtain:

$$\vdash \{\{([\iota\iota a(*\iota A)])\, t\} \bullet [([\iota\iota a(*\overset{\cup}{\iota A})])\, t]\} \Rightarrow$$

$$\Rightarrow \{\{([\iota\iota a(*\iota A)])\, t\} \bullet [([\iota\iota a(*\iota A')])\, t]\} \qquad (271)$$

Now take the axiom of direct attributive restriction (126). Rename iota operators in it to receive:

$$\vdash \{\iota_4 A \Rightarrow \iota_5 A\} \leftrightarrow \{[(\iota_6 A)\iota_4 A] \Rightarrow [(\iota_6 A)\iota_5 A]\} \tag{272}$$

Make the following substitutions (RA_2):

$$\iota_4 A \ \text{ for } \ \iota_4\{\{([\iota\iota a(*\iota A)])\,t\} \bullet [([\iota\iota a(*\iota\overset{\cup}{A})])\,t]\}\,,$$
$$\iota_5 A \ \text{ for } \ \iota_5\{\{([\iota\iota a(*\iota A)])\,t\} \bullet [([\iota\iota a(*\iota A')])\,t]\}\,,$$
$$\iota_6 A \ \text{ for } \ \iota_6\{\iota A\}.$$

Receive:

$$\vdash \underset{3}{\{}\,\iota_4\underset{2}{\{}\,\underset{1}{\{}([\iota\iota a(*\iota A)])\,t\underset{1}{\}} \bullet [([\iota\iota a(*\iota\overset{\cup}{A})])\,t]\underset{2}{\}} \Rightarrow$$
$$\Rightarrow \iota_5\underset{2}{\{}\,\underset{1}{\{}([\iota\iota a(*\iota A)])\,t\underset{1}{\}} \bullet [([\iota\iota a(*\iota A')])\,t]\underset{2}{\}}\,\underset{3}{\}} \leftrightarrow$$
$$\leftrightarrow \underset{3}{\{}\,[(\iota_6\{\iota A\})\iota_4\underset{2}{\{}\,\underset{1}{\{}([\iota\iota a(*\iota A)])\,t\underset{1}{\}} \bullet [([\iota\iota a(*\iota\overset{\cup}{A})])\,t]\underset{2}{\}}] \Rightarrow$$
$$\Rightarrow [(\iota_6\{\iota A\})\iota_5\underset{2}{\{}\,\underset{1}{\{}([\iota\iota a(*\iota A)])\,t\underset{1}{\}} \bullet [([\iota\iota a(*\iota A')])\,t]\underset{2}{\}}]\,\underset{3}{\}} \tag{273}$$

Delete iota operators ι_4, ι_5 and ι_6 on the basis of $RH_{2-1}a$, then apply *MP* to the result and formula (271); receive:

$$\vdash [(\iota A)\underset{2}{\{}\,\underset{1}{\{}([\iota\iota a(*\iota A)])\,t\underset{1}{\}} \bullet [([\iota\iota a(*\iota\overset{\cup}{A})])\,t]\underset{2}{\}}] \Rightarrow$$
$$\Rightarrow [(\iota A)\underset{2}{\{}\,\underset{1}{\{}([\iota\iota a(*\iota A)])\,t\underset{1}{\}} \bullet [([\iota\iota a(*\iota A')])\,t]\underset{2}{\}}] \tag{274}$$

Accordingly to the rule RJ_2 replace the closed antecedent and consequent in (274) by open formulae:

$$\vdash \{(\iota A)\underset{2}{\{}\,\underset{1}{\{}([\iota\iota a(*\iota A)])\,t\underset{1}{\}} \bullet [([\iota\iota a(*\iota\overset{\cup}{A})])\,t]\underset{2}{\}}\} \Rightarrow$$
$$\Rightarrow \{(\iota A)\underset{2}{\{}\,\underset{1}{\{}([\iota\iota a(*\iota A)])\,t\underset{1}{\}} \bullet [([\iota\iota a(*\iota A')])\,t]\underset{2}{\}}\} \tag{275}$$

The antecedent of (275) coincides with the definiens of non-minimal system, and the consequent with the definiens of non-unique system. Theorem 5 is proven.

4.6. ***Theorem* 6**. Any substratum-open system is a non-unique one.

The proof. Take the definition (16) of the object $\iota\overset{\Delta}{A}$:

$$\iota\overset{\Delta}{A} =_{def} [(\iota A')\{\iota A' \supset \iota A\}]$$

According to the rule RF_1 this definition implies that the following formula is accepted:

$$\vdash \iota\overset{\Delta}{A} \Rightarrow \iota A'$$

Analogous to the proof of the previous theorem, we can deduce from it the following:

$$\vdash \{(\iota A)\underset{2}{\{}\,\underset{1}{\{}([\iota\iota a(*\iota A)])\,t\underset{1}{\}} \bullet [([\iota\iota a(*\iota\overset{\Delta}{A})])\,t]\underset{2}{\}}\} \Rightarrow$$

$$\Rightarrow \{(\iota A)\{_2\ \{_1([\iota\iota a(*\iota A)])\,t\}_1 \bullet [([\iota\iota a(*\iota A')])\,t]\ \}_2\}$$

Here the antecedent of the implication coincides with the definiens of substratum-open system, and its consequent coincides with the definiens of non-unique system, so the theorem is proven.

4.7. ***Theorem*** **7**. If a system is (1) totalitarian and (2) structurally non-variable, then it is also a rigid one.

The proof. Applying the rule RF_2 to the definitions of totalitarian (30) and structurally non-variable (31) systems, we obtain the definition of a system that possesses both properties:

(ιA)Totalitarian and structurally non-variable system $=_{def}$

$$(\iota A)\{\ \{(\ [\ a(*\iota A)]\ )t\} \bullet \{\ t \to [A(*\iota A)]\} \bullet \bullet\{(\ [\iota\iota a(*\iota A)]\ )t\} \bullet \{\ [A(*\iota A)] \Rightarrow \iota\iota a\}\ \} \qquad (276)$$

The first component of the related list in the definiens of (276) is the consequence of the third component due to the rule $RH_{2-1}a$. Therefore, with the help of RC_5, we can delete it from the list. Then put the component $\{([\iota\iota a\ (*\iota A)])t\}$ of the list onto its first place, and bind the free occurrences of A in all components of the list with iota operator $\iota\iota$, which is possible due to RH_9. We obtain the final definition:

(ιA)Totalitarian and structurally non-variable system $=_{def}$

$$(\iota A)\{\ \{(\ [\iota\iota a(*\iota A)]\ )t\} \bullet \{t \to [\iota\iota A(*\iota A)]\} \bullet \{\ [\iota\iota A(*\iota A)] \Rightarrow \iota\iota a\}\ \} \qquad (277)$$

Use the axiom of transitivity for neutral and attributive implication (103) and rename iota operators in it (RH_3) to obtain

$$\vdash \{\{\iota_4 A \to \iota_5 A\} \bullet \{\iota_5 A \Rightarrow \iota_6 A\}\} \to \{\iota_4 A \to \iota_6 A\} \qquad (278)$$

Make the following substitutions (RA_2) in (278): $\iota_4 A$ for $\iota_4 t$, $\iota_5 A$ for $\iota_5 [\iota\iota A(*\iota A)]$ and $\iota_6 A$ for $\iota_6 \{\iota\iota a\}$. Obtain:

$$\vdash \{\{\iota_4 t \to \iota_5 [\iota\iota A(*\iota A)]\} \bullet \{\iota_5 [\iota\iota A(*\iota A)] \Rightarrow \iota_6 \{\iota\iota a\}\}\} \to \{\iota_4 t \to \iota_6 \{\iota\iota a\}\} \qquad (279)$$

Omit iota operators ι_4 , ι_5 and ι_6 on the basis of the rule $RH_{2-1}a$. Obtain:

$$\vdash \{\{t \to [\iota\iota A(*\iota A)]\} \bullet \{[\iota\iota A(*\iota A)] \Rightarrow \iota\iota a\}\} \to \{\ t \to \iota\iota a\} \qquad (280)$$

Take now the axiom (89) with $\Rightarrow$ as $\iota\iota\!\to$. Rename iota operators in it (RH_3) to obtain:

$$\vdash \{\{(\iota_4 A \bullet \iota_5 A)\mathrm{T}\} \bullet \{\iota_5 A \Rightarrow \iota_6 A\}\} \to \{\iota_4 A \bullet \iota_6 A\} \qquad (281)$$

Make the following substitutions (RA_2) in (281):

$\iota_4 A$ for $\iota_4 \{(\ [\iota\iota a(*\iota A)]\ )t\}$,

$\iota_5 A$ for $\iota_5 \{\{t \to [\iota\iota A(*\iota A)]\} \bullet \{[\iota\iota A(*\iota A)] \Rightarrow \iota\iota a\}\}$,

$\iota_6 A$ for $\iota_6 \{t \to \iota\iota a\}$.

Obtain:

$$\vdash \underset{4}{\{}\underset{3}{\{}\underset{2}{(}\iota_4\underset{1}{\{}([\iota\iota a(*\iota A)])t\underset{1}{\}}\bullet\iota_5\underset{2}{\{}\underset{1}{\{}t\to[\iota\iota\iota A(*\iota A)]\underset{1}{\}}\bullet\underset{1}{\{}[\iota\iota\iota A(*\iota A)]\Rightarrow\iota\iota a\underset{1}{\}}\underset{2}{\}}\underset{2}{)}\mathrm{T}\underset{3}{\}}\bullet$$

$$\bullet\underset{3}{\{}\iota_5\underset{2}{\{}\underset{1}{\{}t\to[\iota\iota\iota A(*\iota A)]\underset{1}{\}}\bullet\underset{1}{\{}[\iota\iota\iota A(*\iota A)]\Rightarrow\iota\iota a\underset{1}{\}}\underset{2}{\}}\Rightarrow\iota_6\{t\to\iota\iota a\}\underset{3}{\}}\underset{4}{\}}\to$$

$$\to\underset{2}{\{}\iota_4\underset{1}{\{}([\iota\iota a(*\iota A)])t\underset{1}{\}}\bullet\iota_6\{t\to\iota\iota a\}\underset{2}{\}} \qquad (282)$$

Omit iota operators ι_4 , ι_5 and ι_6 on the basis of the rule $RH_{2-1}a$. Receive:

$$\vdash \underset{4}{\{}\underset{3}{\{}\underset{2}{(}\underset{1}{\{}([\iota\iota a(*\iota A)])t\underset{1}{\}}\bullet\underset{2}{\{}\underset{1}{\{}t\to[\iota\iota\iota A(*\iota A)]\underset{1}{\}}\bullet\underset{1}{\{}[\iota\iota\iota A(*\iota A)]\Rightarrow\iota\iota a\underset{1}{\}}\underset{2}{\}}\underset{2}{)}\mathrm{T}\underset{3}{\}}\bullet$$

$$\bullet\underset{3}{\{}\underset{2}{\{}\underset{1}{\{}t\to[\iota\iota\iota A(*\iota A)]\underset{1}{\}}\bullet\underset{1}{\{}[\iota\iota\iota A(*\iota A)]\Rightarrow\iota\iota a\underset{1}{\}}\underset{2}{\}}\Rightarrow\{t\to\iota\iota a\}\underset{3}{\}}\underset{4}{\}}\to$$

$$\to\underset{2}{\{}\underset{1}{\{}([\iota\iota a(*\iota A)])t\underset{1}{\}}\bullet\{t\to\iota\iota a\}\underset{2}{\}} \qquad (283)$$

Now use the axiom (127) and rename iota operators in it (RH_3) to obtain

$$\vdash \{\iota_4 A\to\iota_5 A\}\leftrightarrow\{[(\iota_6 A)\iota_4 A]\to[(\iota_6 A)\iota_5 A]\} \qquad (284)$$

Make the following substitutions (RA_2) in (284):

$\iota_4 A$ for the antecedent of the implication (283) preceded by ι_4 , i.e.:

$$\iota_4\underset{4}{\{}\underset{3}{\{}\underset{2}{(}\underset{1}{\{}([\iota\iota a(*\iota A)])t\underset{1}{\}}\bullet\underset{2}{\{}\underset{1}{\{}t\to[\iota\iota\iota A(*\iota A)]\underset{1}{\}}\bullet\underset{1}{\{}[\iota\iota\iota A(*\iota A)]\Rightarrow\iota\iota a\underset{1}{\}}\underset{2}{\}}\underset{2}{)}\mathrm{T}\underset{3}{\}}\bullet$$

$$\bullet\underset{3}{\{}\underset{2}{\{}\underset{1}{\{}t\to[\iota\iota\iota A(*\iota A)]\underset{1}{\}}\bullet\underset{1}{\{}[\iota\iota\iota A(*\iota A)]\Rightarrow\iota\iota a\underset{1}{\}}\underset{2}{\}}\Rightarrow\{t\to\iota\iota a\}\underset{3}{\}}\underset{4}{\}} \qquad (285)$$

$\iota_5 A$ for the consequent of (283) preceded by ι_5 , i.e.:

$$\iota_5\underset{2}{\{}\underset{1}{\{}([\iota\iota a(*\iota A)])t\underset{1}{\}}\bullet\{t\to\iota\iota a\}\underset{2}{\}} \qquad (286)$$

$$\iota_6 A \text{ for } \iota_6\{\iota A\} \qquad (287)$$

After these substitutions the rule $RH_{2-1}a$ will be applicable, so we can delete iota operators ι_4 , ι_5 and ι_6 from the result. Thus we obtain the implication, whose antecedent, i.e. (283), has been proved. Applying *MP*, we obtain the formula thus obtained from the consequent of (284), i.e. from

$$[(\iota_6 A)\iota_4 A]\to[(\iota_6 A)\iota_5 A] \qquad (288)$$

should be accepted. On the basis of the rule RJ_2 we can state that the implication

$$\{(\iota_6 A)\iota_4 A\}\to\{(\iota_6 A)\iota_5 A\} \qquad (289)$$

obtained (after the above transformations) from (288) by opening its antecedent and consequent should be accepted, too:

$$\vdash\{(\iota A)\underset{4}{\{}\underset{3}{\{}\underset{2}{(}\underset{1}{\{}([\iota\iota a(*\iota A)])t\underset{1}{\}}\bullet\underset{2}{\{}\underset{1}{\{}t\to[\iota\iota\iota A(*\iota A)]\underset{1}{\}}\bullet\underset{1}{\{}[\iota\iota\iota A(*\iota A)]\Rightarrow\iota\iota a\underset{1}{\}}\underset{2}{\}}\underset{2}{)}\mathrm{T}\underset{3}{\}}\bullet$$

$$\bullet\underset{3}{\{}\underset{2}{\{}\underset{1}{\{}t\to[\iota\iota\iota A(*\iota A)]\underset{1}{\}}\bullet\underset{1}{\{}[\iota\iota\iota A(*\iota A)]\Rightarrow\iota\iota a\underset{1}{\}}\underset{2}{\}}\Rightarrow\{t\to\iota\iota a\}\underset{3}{\}}\underset{4}{\}}\}\to$$

$$\to\{(\iota A)\underset{2}{\{}\underset{1}{\{}([\iota\iota a(*\iota A)])t\underset{1}{\}}\bullet\{t\to\iota\iota a\}\underset{2}{\}}\} \qquad (290)$$

The consequence of (290) coincides with the definiens of the rigid system. What about the antecedent? From the formal point of view there are two distinctions of the (290) antecedent from the definiens of the totalitarian and structurally non-variable system (277). The first is the presence of the valent sign T in the middle of the formula, but we may omit it due to

RF_4. The second distinction is the presence of the following neutral implication in the (290) antecedent:

$$\{_2\{_1\, t\to[\iota\iota A(*\iota A)]\,\}_1 \bullet \{_1[\iota\iota A(*\iota A)]\Rightarrow\iota\iota a\}_1\}_2 \Rightarrow \{t\to\iota\iota a\} \qquad (291)$$

This implication can be deduced from the axiom (103) by making necessary substitutions in it and then deleting some iota operators that is possible on the basis of the rule $RH_{2-1}a$.

According to the rule RF_3 the provable component of the list, that is (291), may be omitted from the definiens of the definition. Therefore, we obtain from (290):

$$\vdash \{(\iota A)\{_2\{_1([\iota\iota a(*\iota A)])t\}_1 \bullet \{_1 t\to[\iota\iota A(*\iota A)]\}_1 \bullet \{_1[\iota\iota A(*\iota A)]\Rightarrow\iota\iota a\}_1\}_2\} \to$$

$$\to \{(\iota A)\{_2\,\{_1([\iota\iota a(*\iota A)]\,)t\}_1 \bullet \{t\to\iota\iota a\}\,\}_2\}$$

Ergo, theorem 7 is proven.

During the proof of system-theoretical laws we have used only a part of the TDL resources. The usage of other resources permits us to enlarge the number of parametrical GST theorems.

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